Counting and Sampling Solutions of SAT/SMT Constraints

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[Extended version of slides presented at SAT/SMT/AR Summer School 2016, Lisbon]

Problem Definition

- Given
  - X_1 , … X_n : variables with finite discrete domains D_1, … D_n
  - Constraint (logical formula) F over X_1 , … X_n
  - Weight function W: $D_1 \times \ldots D_n \rightarrow \Re^{\geq 0}$

Let R_F: set of assignments of X_1 , … X_n that satisfy F

- Determine $W(R_F) = \sum_{y \in R_F} W(y)$
  If W(y) = 1 for all y, then $W(R_F) = |R_F|$

Discrete Integration (Model Counting)

- Randomly sample from R_F such that Pr[y is sampled] $\propto$ W(y)
  If W(y) = 1 for all y, then uniformly sample from R_F

Discrete Sampling

Suffices to consider all domains as {0, 1}: assume for this tutorial

Discrete Integration: An Application

- **Probabilistic Inference**
  - An **alarm** rings if it's in a working state when an **earthquake** happens or a **burglary** happens
  - The **alarm** can malfunction and ring without **earthquake** or **burglary** happening
  - Given that the **alarm** rang, what is the likelihood that an **earthquake** happened?
  - Given conditional dependencies (and conditional probabilities) calculate **Pr[event | evidence]**
    - What is **Pr [Earthquake | Alarm]** ?

Discrete Integration: An Application

Probabilistic Inference: Bayes' rule to the rescue

$$\Pr[event_i \mid evidence] = \frac{\Pr[event_i \cap evidence]}{\Pr[evidence]} = \frac{\Pr[event_i \cap evidence]}{\sum_j \Pr[event_j \cap evidence]}$$

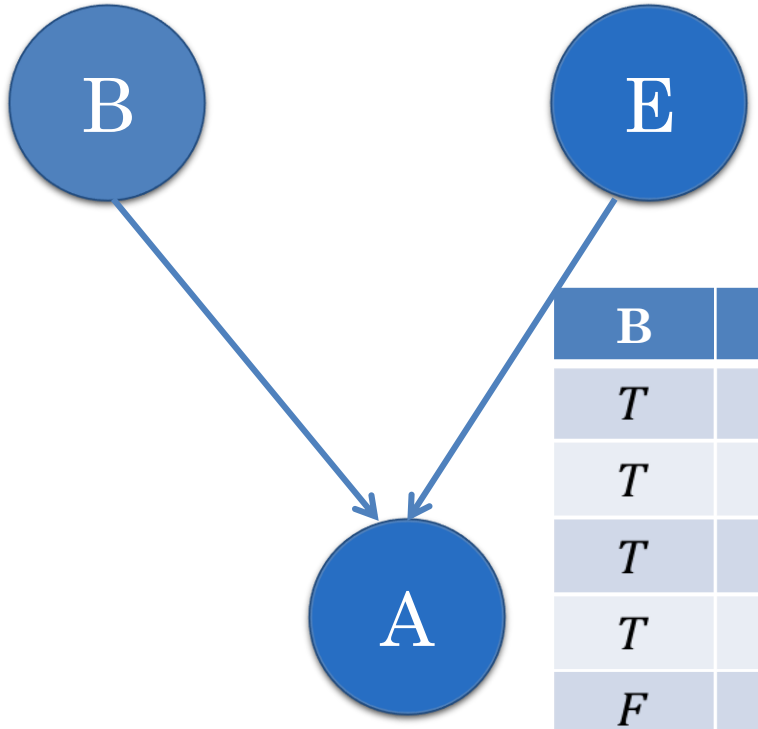
$$\Pr[event_j \cap evidence] = \Pr[evidence \mid event_j] \times \Pr[event_j]$$

How do we represent conditional dependencies efficiently, and calculate these probabilities?

Discrete Integration: An Application

Probabilistic Graphical Models

B	Pr
T	0.8
F	0.2



E	Pr
T	0.1
F	0.9

B	E	A	Pr(A | E,B)
T	T	T	0.3
T	T	F	0.7
T	F	T	0.4
T	F	F	0.6
F	T	T	0.2
F	F	F	0.8
F	F	T	0.1
F	F	F	0.9

Conditional Probability Tables (CPT)

Discrete Integration: An Application

B	Pr
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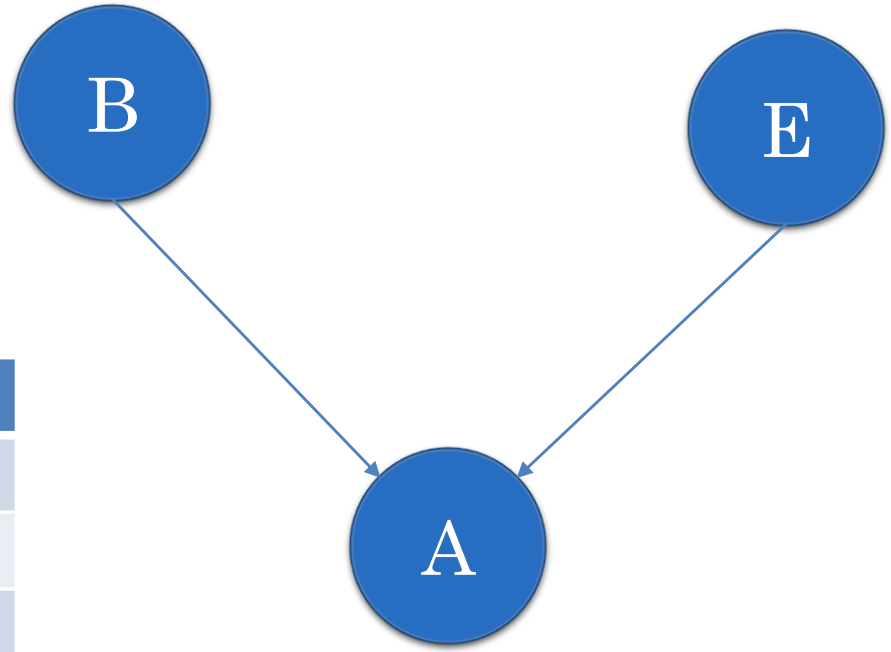
F	0.2



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F	T	T	0.2
F	F	F	0.8
F	F	T	0.1
F	F	F	0.9

$$\Pr[E \cap A]$$
$$= \Pr[E] * \Pr[\neg B] * \Pr[A|E, \neg B]$$
$$+\Pr[E] * \Pr[B] * \Pr[A|E, B]$$

Discrete Integration: An Application

- **Probabilistic Inference: From probabilities to logic**

$V = \{v_A, v_{\sim A}, v_B, v_{\sim B}, v_E, v_{\sim E}\}$ Prop vars corresponding to events

$T = \{t_{A|B,E}, t_{\sim A|B,E}, t_{A|B,\sim E} \ldots\}$ Prop vars corresponding to CPT entries

Formula encoding probabilistic graphical model (φ_{PGM}):

$(v_A \oplus v_{\sim A}) \wedge (v_B \oplus v_{\sim B}) \wedge (v_E \oplus v_{\sim E})$ Exactly one of v_A and $v_{\sim A}$ is true

$\wedge$

$(t_{A|B,E} \Leftrightarrow v_A \wedge v_B \wedge v_E) \wedge (t_{\sim A|B,E} \Leftrightarrow v_{\sim A} \wedge v_B \wedge v_E) \wedge \ldots$

If v_A , v_B , v_E are true, so must $t_{A|B,E}$ and vice versa

Discrete Integration: An Application

- **Probabilistic Inference: From probabilities to logic and weights**

$V = \{v_A, v_{\sim A}, v_B, v_{\sim B}, v_E, v_{\sim E}\}$

$T = \{t_{A|B,E}, t_{\sim A|B,E}, t_{A|B,\sim E} \ldots\}$

$W(v_{\sim B}) = 0.2, W(v_B) = 0.8$ Probabilities of indep events are weights of +ve literals

$W(v_{\sim E}) = 0.1, W(v_E) = 0.9$

$W(t_{A|B,E}) = 0.3, W(t_{\sim A|B,E}) = 0.7, \ldots$ CPT entries are weights of +ve literals

$W(v_A) = W(v_{\sim A}) = 1$ Weights of vars corresponding to dependent events

$W(\neg v_{\sim B}) = W(\neg v_B) = W(\neg t_{A|B,E}) \ldots = 1$ Weights of -ve literals are all 1

Weight of assignment $(v_A = 1, v_{\sim A} = 0, t_{A|B,E} = 1, \ldots) = W(v_A) * W(\neg v_{\sim A}) * W(t_{A|B,E}) * \ldots$

Product of weights of literals in assignment

Discrete Integration: An Application

- **Probabilistic Inference: From probabilities to logic and weights**

$V = \{v_A, v_{\sim A}, v_B, v_{\sim B}, v_E, v_{\sim E}\}$

$T = \{t_{A|B,E}, t_{\sim A|B,E}, t_{A|B,\sim E} \ldots\}$

Formula encoding combination of events in probabilistic model

(Alarm and Earthquake) $F = \varphi_{PGM} \wedge v_A \wedge v_E$

Set of satisfying assignments of F:

$R_F = \{ (v_A = 1, v_E = 1, v_B = 1, t_{A|B,E} = 1, \text{all else } 0), (v_A = 1, v_E = 1, v_{\sim B} = 1, t_{A|\sim B,E} = 1, \text{all else } 0) \}$

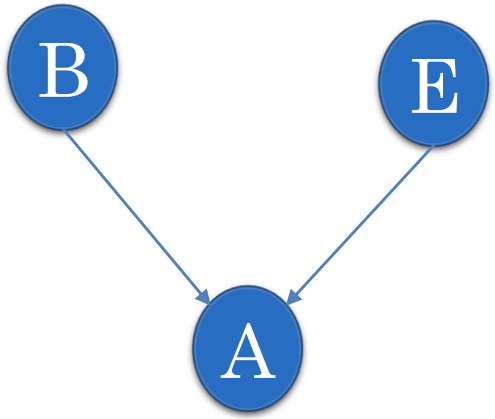
Weight of satisfying assignments of F:

$$W(R_F) = W(v_A) * W(v_E) * W(v_B) * W(t_{A|B,E}) + W(v_A) * W(v_E) * W(v_{\sim B}) * W(t_{A|\sim B,E})$$

$$= 1 * Pr[E] * Pr[B] * Pr[A \mid B,E] + 1 * Pr[E] * Pr[\sim B] * Pr[A \mid \sim B,E] = Pr[A \cap E]$$

Discrete Integration: An Application

From probabilistic inference to unweighted model counting



$\Pr[E|A]$

Roth 1996

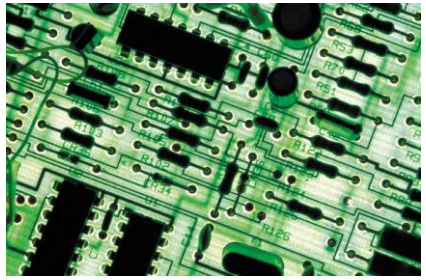
Weighted Model Counting

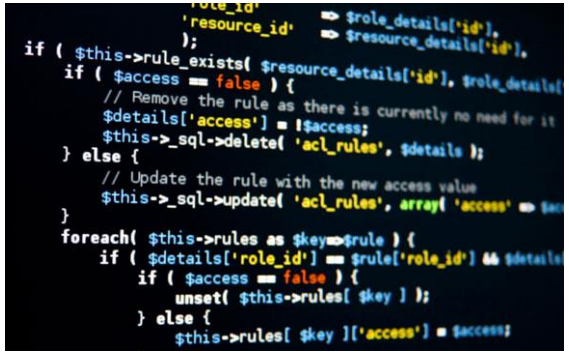
Weighted Model Counting ⇒ Unweighted Model Counting

IJCAI 2015

Reduction polynomial in #bits representing CPT entries

Discrete Sampling: An Application





Functional Verification

- Formal verification
  - Challenges: formal requirements, scalability
  - ~10-15% of verification effort
- Dynamic verification: ***dominant approach***

Discrete Sampling: An Application

- Design is simulated with test vectors
- Test vectors represent different verification scenarios
- Results from simulation compared to intended results

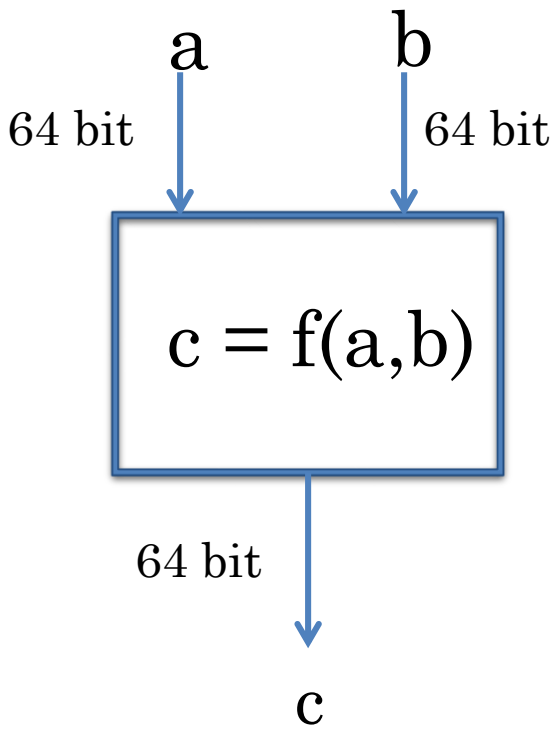
- How do we generate test vectors?

Challenge: Exceedingly large test input space!

Can't try all input combinations

2^{128} combinations for a 64-bit binary operator!!!

Discrete Sampling: An Application

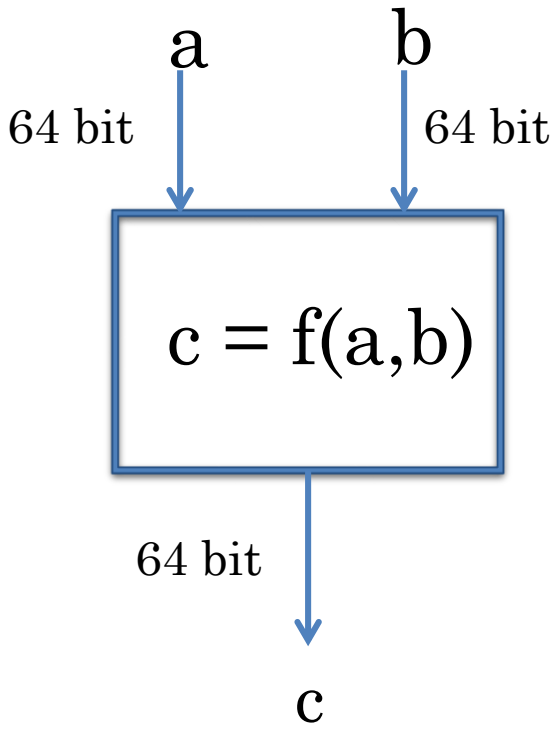


Sources for Constraints

- Designers:
  1. $a +_{64} 11 *_{32} b = 12$
  2. $a <_{64} (b >> 4)$
- Past Experience:
  1. $40 <_{64} 34 + a <_{64} 5050$
  2. $120 <_{64} b <_{64} 230$
- Users:
  1. $232 *_{32} a + b \mathrel{!}= 1100$
  2. $1020 <_{64} (b /_{64} 2) +_{64} a <_{64} 2200$

- Test vectors: solutions of constraints

Discrete Sampling: An Application



Constraints

- Designers:
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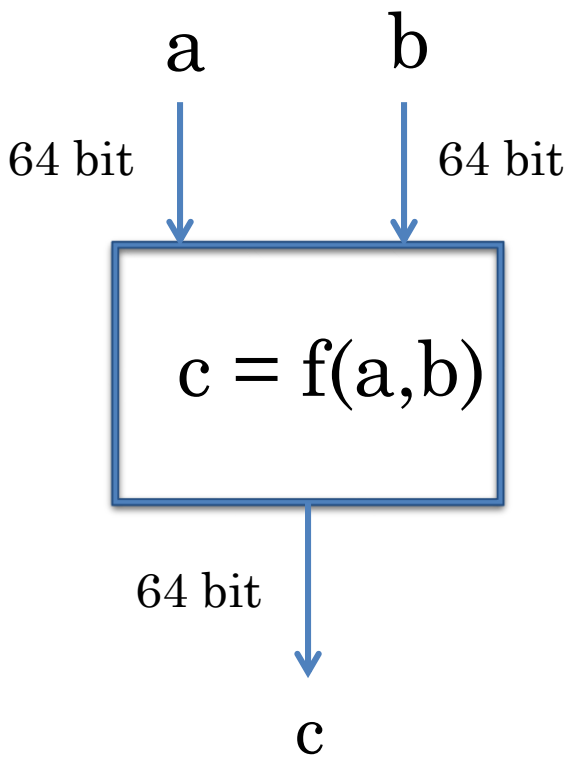
Modern SAT/SMT solvers are complex systems

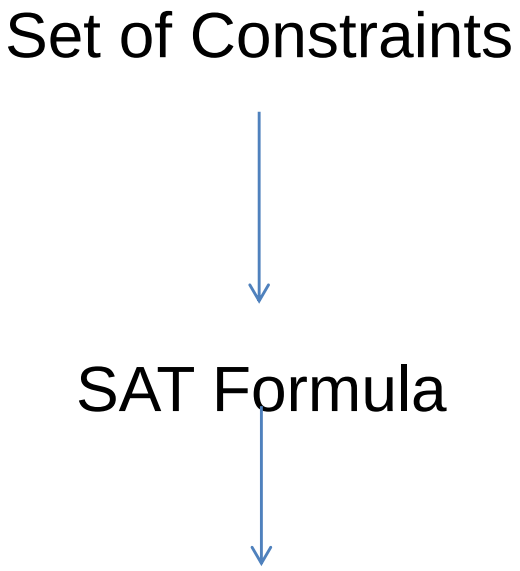
Efficiency stems from the solver automatically "biasing" search

Fails to give unbiased or user-biased distribution of test vectors

Discrete Sampling: An Application

Constrained Random Verification





Sample satisfying assignments uniformly at random

Scalable Uniform Generation of SAT Witnesses

Discrete Integration and Sampling

- Many, many more applications
  - Physics, economics, network reliability estimation, …
- Discrete integration and discrete sampling are closely related
  - Insights into solving one efficiently and approximately can often be carried over to solving the other
  - More coming in subsequent slides …

Agenda (Part I)

- Hardness of counting/integration and sampling
- Early work on counting and sampling
- Universal hashing
- Universal-hashing based algorithms: an overview

How Hard is it to Count/Sample?

- Trivial if we could enumerate R_F: Almost always impractical
- Computational complexity of counting (discrete integration):

Exact unweighted counting: #P-complete [Valiant 1978]

Approximate unweighted counting:

Deterministic: Polynomial time det. Turing Machine with Σ_2^p oracle [Stockmeyer 1983]

$$\frac{|R_F|}{1+\varepsilon} \leq \text{DetEstimate}(\text{F}, \varepsilon) \leq |R_F| \times (1+\varepsilon), \text{ for } \varepsilon > 0$$

Randomized: Polynomial time probabilistic Turing Machine with NP oracle

[Stockmeyer 1983; Jerrum,Valiant,Vazirani 1986]

$$\Pr\left[\frac{|R_F|}{1+\varepsilon} \leq \text{RandEstimate}(\text{F}, \varepsilon, \delta) \leq |R_F| \cdot (1+\varepsilon)\right] \geq 1-\delta, \text{ for } \varepsilon > 0, \ 0 < \delta \leq 1$$

Probably Approximately Correct (PAC) algorithm

Weighted versions of counting: Exact: #P-complete [Roth 1996],

Approximate: same class as unweighted version [follows from Roth 1996]

How Hard is it to Count/Sample?

- Computational complexity of sampling:

Uniform sampling: Polynomial time prob. Turing Machine with NP oracle
[Bellare,Goldreich,Petrank 2000]

$$\Pr[y = \text{UniformGenerator(F)}] = c, \text{ where } \begin{cases} c = 0 \text{ if } y \notin \mathrm{R_F} \\ c > 0 \text{ and indep of } y \text{ if } y \in \mathrm{R_F} \end{cases}$$

Almost uniform sampling: Polynomial time prob. Turing Machine with NP oracle
[Jerrum,Valiant,Vazirani 1986, also from Bellare,Goldreich,Petrank 2000]

$$\frac{c}{1+\varepsilon} \le \Pr[y = \text{AUGenerator(F}, \varepsilon)] \le c \cdot (1+\varepsilon), \text{ where } \begin{cases} c = 0 \text{ if } y \notin \mathrm{R_F} \\ c > 0 \text{ and indep of } y \text{ if } y \in \mathrm{R_F} \end{cases}$$

Pr[Algorithm outputs some y] ≥ ½, if F is satisfiable

Exact Counters

- DPLL based counters [CDP: Birnbaum,Lozinski 1999]
  - DPLL branching search procedure, with partial truth assignments
  - Once a branch is found satisfiable, if t out of n variables assigned, add 2^{n-t} to model count, backtrack to last decision point, flip decision and continue
  - Requires data structure to check if all clauses are satisfied by partial assignment

    Usually not implemented in modern DPLL SAT solvers
  - Can output a lower bound at any time

Exact Counters

- DPLL + component analysis [RelSat: Bayardo, Pehoushek 2000]
  - Constraint graph G:
    Variables of F are vertices
    An edge connects two vertices if corresponding variables appear in some clause of F
  - Disjoint components of G lazily identified during DPLL search
  - F1, F2, … Fn : subformulas of F corresponding to components
    $|R_F| = |R_{F1}| * |R_{F2}| * |R_{F3}| * \ldots$
  - Heuristic optimizations:
    Solve most constrained sub-problems first
    Solving sub-problems in interleaved manner

Exact Counters

- DPLL + Caching [Bacchus et al 2003, Cachet: Sang et al 2004, sharpSAT: Thurley 2006]

  If same sub-formula revisited multiple times during DPLL search, cache result and re-use it

  “Signature” of the satisfiable sub-formula/component must be stored

  Different forms of caching used:

    Simple sub-formula caching

    Component caching

    Linear-space caching

  Component caching can also be combined with clause learning and other easoning techniques at each node of DPLL search tree

    WeightedCachet: DPLL + Caching for weighted assignments

Exact Counters

- Knowledge Compilation based
  - Compile given formula to another form which allows counting models in time polynomial in representation size
  - Reduced Ordered Binary Decision Diagrams (ROBDD) [Bryant 1986]: Construction can blow up exponentially
  - Deterministic Decomposable Negation Normal Form (d-DNNF) [c2d: Darwiche 2004]

    Generalizes ROBDDs; can be significantly more succinct

    Negation normal form with following restrictions:

    Decomposability: All AND operators have arguments with disjoint support

    Determinizability: All OR operators have arguments with disjoint solution sets
  - Sentential Decision Diagrams (SDD) [Darwiche 2011]

Exact Counters: How far do they go?

- Work reasonably well in small-medium sized problems, and in large problem instances with special structure
- Use them whenever possible
  - #P-completeness hits back eventually – scalability suffers!

Bounding Counters

[MBound: Gomes et al 2006; SampleCount: Gomes et al 2007; BPCount: Kroc et al 2008]

- Provide lower and/or upper bounds of model count
- Usually more efficient than exact counters
- No approximation guarantees on bounds

  Useful only for limited applications

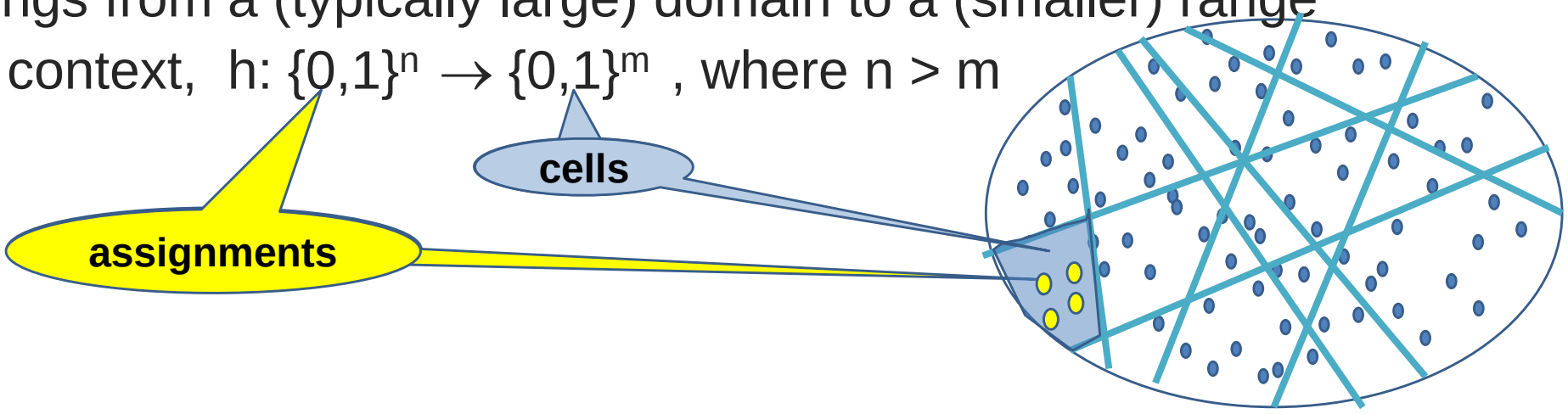
Markov Chain Monte Carlo Techniques

- Rich body of theoretical work with applications to sampling and counting [Jerrum,Sinclair 1996]
- Some popular (and intensively studied) algorithms:
  - Metropolis-Hastings [Metropolis et al 1953, Hastings 1970], Simulated Annealing [Kirkpatrick et al 1982]
- High-level idea:
  - Start from a “state” (assignment of variables)
  - Randomly choose next state using “local” biasing functions (depends on target distribution & algorithm parameters)
  - Repeat for an appropriately large number (N) of steps
  - After N steps, samples follow target distribution with high confidence
- Convergence to desired distribution guaranteed only after N (large) steps
- In practice, steps truncated early heuristically

  Nullifies/weakens theoretical guarantees [Kitchen,Kuehlman 2007]

Hashing-based Sampling/Counting

- Extremely successful in recent years [CP2013, CAV2013, NIPS2013, DAC 2014, AAAI 2014, UAI 2014, NIPS 2014, ICML 2014, UAI 2015, ICML 2015, AAAI 2016, ICML 2016, IJCAI 2016, …]
- Focus of remainder of tutorial
- Hash functions:
  - Mappings from a (typically large) domain to a (smaller) range
  - In our context, $h: \{0,1\}^n \rightarrow \{0,1\}^m$, where $n > m$



More on Hash Functions

- Good deterministic hash function:
  - Inputs distributed uniformly $\Rightarrow$ All cells are small in expectation
  - But solutions of constraints can't be considered random
- Universal hash functions [Carter,Wegman 1977; Sipser 1983]
  - Define a family of hash functions H having some properties
    Each $h \in H$ is a function: $\{0,1\}^n \rightarrow \{0,1\}^m$
  - Choose randomly one hash function h from H
  - For every distribution of inputs, all cells are small and similar in expectation
    Guarantees probabilistic properties of cell sizes even without knowing distribution of inputs
  - Used by Sipser (1983) for combinatorial optimization, by Stockmeyer (1983) for deterministic approximate counting

Universality of Hash Functions and Complexity

- H(n,m,r): Family of r-universal hash functions
  - $h : \{0,1\}^n \rightarrow \{0,1\}^m$
  - For every $X \in \{0,1\}^n$ and every $\alpha \in \{0,1\}^m$

    $\Pr[\, h(X) = \alpha \mid h \text{ chosen uniformly rand. from } H \,] = 1/2^m$

    Uniformity
  - For distinct $X_1, \ldots X_r \in \{0,1\}^n$ and for every $\alpha_1, \ldots \alpha_r \in \{0,1\}^m$,

    $\Pr[h(X_1) = \alpha_1 \wedge \ldots \wedge h(X_r) = \alpha_r \mid h \text{ rand. From } H \,] = 1/2^{mr}$.

    Independence-like
- Higher r $\Rightarrow$ Stronger guarantees on size of cells

  Lower probability of large variations in cell sizes
- r-wise universality can be implemented using polynomials of degree r-1 in $GF(2^{\max(n,m)})$

  Can be computationally challenging; say n = r = 10000, m < n
- Lower r $\Rightarrow$ Lower complexity of reasoning about r-universal hashing

2-Universal Hashing: Simple to Compute

- Variables: $X_1, X_2, X_3, \ldots, X_n$
- To construct h: $\{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- Pick every variable with prob. ½ , XOR them and add 1 with prob. ½
- E.g.: $X_1 \oplus X_3 \oplus X_6 \oplus \ldots \oplus X_{n-1}$
- $\alpha \in \{0,1\}^m \rightarrow$ Set every XOR equation to 0 or 1 randomly
- The cell: FΛXOR (CNF+XOR)

$$
\begin{aligned}
&X_1 \oplus X_3 \oplus X_6 \oplus \ldots X_{n-1} = 0 \\
&X_1 \oplus X_2 \oplus X_4 \oplus \ldots X_{n-1} = 1 \\
&X_1 \oplus X_3 \oplus X_5 \oplus \ldots X_{n-1} = 0 \\
&X_2 \oplus X_3 \oplus X_4 \oplus \ldots X_{n-1} = 0 \\
&\ldots\ldots \\
&X_1 \oplus X_2 \oplus X_3 \oplus \ldots X_{n-1} = 0
\end{aligned}
\quad \Big\} \; m \text{ XORs}
$$

2-Universal Hashing: Yet Powerful

- Let X be the number of solutions of F in an arbitrarily chosen cell
  - What is μ_X, and how much can X deviate from μ_X?
- For every $y \in R_F$, we define $\mathrm{I}_y = \begin{cases} 1, & y \text{ is in cell} \\ 0, & \text{otherwise} \end{cases}$
- $X = \sum_{y \in R_F} I_y$
  - $\mu_X = \frac{|R_F|}{2^m}$..... From random choice of hash function
  - $\sigma_X^2 \leq \mu_X$..... From 2-universality of hash function
- This gives the concentration bound:

$$\Pr\left[\frac{\mu_X}{1+\epsilon} \leq X \leq \mu_X(1+\epsilon)\right] \geq 1 - \frac{\sigma^2}{(\frac{\varepsilon}{1+\epsilon})^2(\mu_X)^2} \geq 1 - \frac{1}{(\frac{\varepsilon}{1+\epsilon})^2\mu_X}$$

Having $\mu_X > k(1+\frac{1}{\epsilon^2})$ gives us $1 - \frac{1}{k}$ lower bound

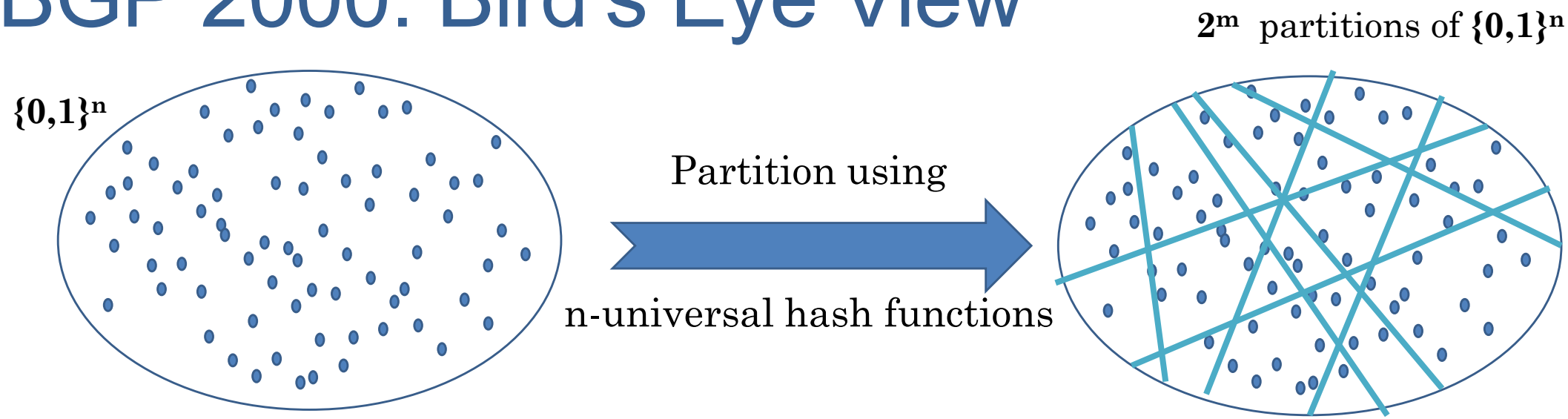
Hashing-based Sampling

- Bellare, Goldreich, Petrank (BGP 2000)
  - Uniform generator for SAT witnesses:
    - Polynomial time randomized algorithm with access to an NP oracle

$$\Pr[y = \mathrm{BGP}(\mathrm{F})] = \begin{cases} 0 \text{ if } y \notin \mathrm{R}_{\mathrm{F}} \\ c\ (>0) \text{ if } y \in \mathrm{R}_{\mathrm{F}}, \text{ where } c \text{ is independent of } y \end{cases}$$

  - Employs n-**universal hash functions**
    - Works well for small values of n
    - For high dimensions (large n), significant computational overheads

BGP 2000: Bird's Eye View

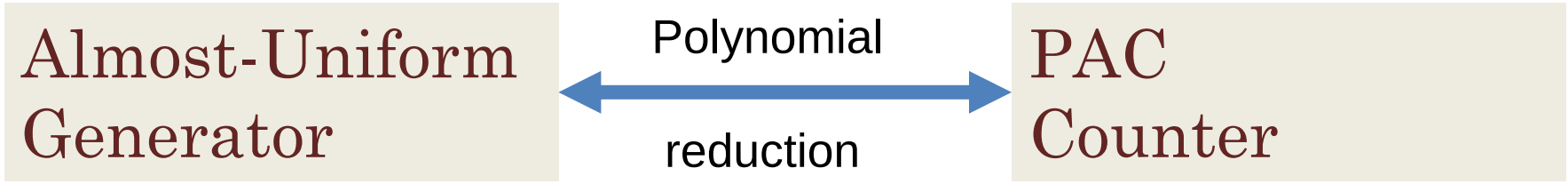


- For right choice of m, all the cells are small (# of solutions $\leq 2n^2$)
- Check if all the cells are small (NP- Query)
- If yes, pick a solution randomly from randomly p cked cell

In practice, the query is too long and complex for large n, and can not be handled by modern SAT Solvers!

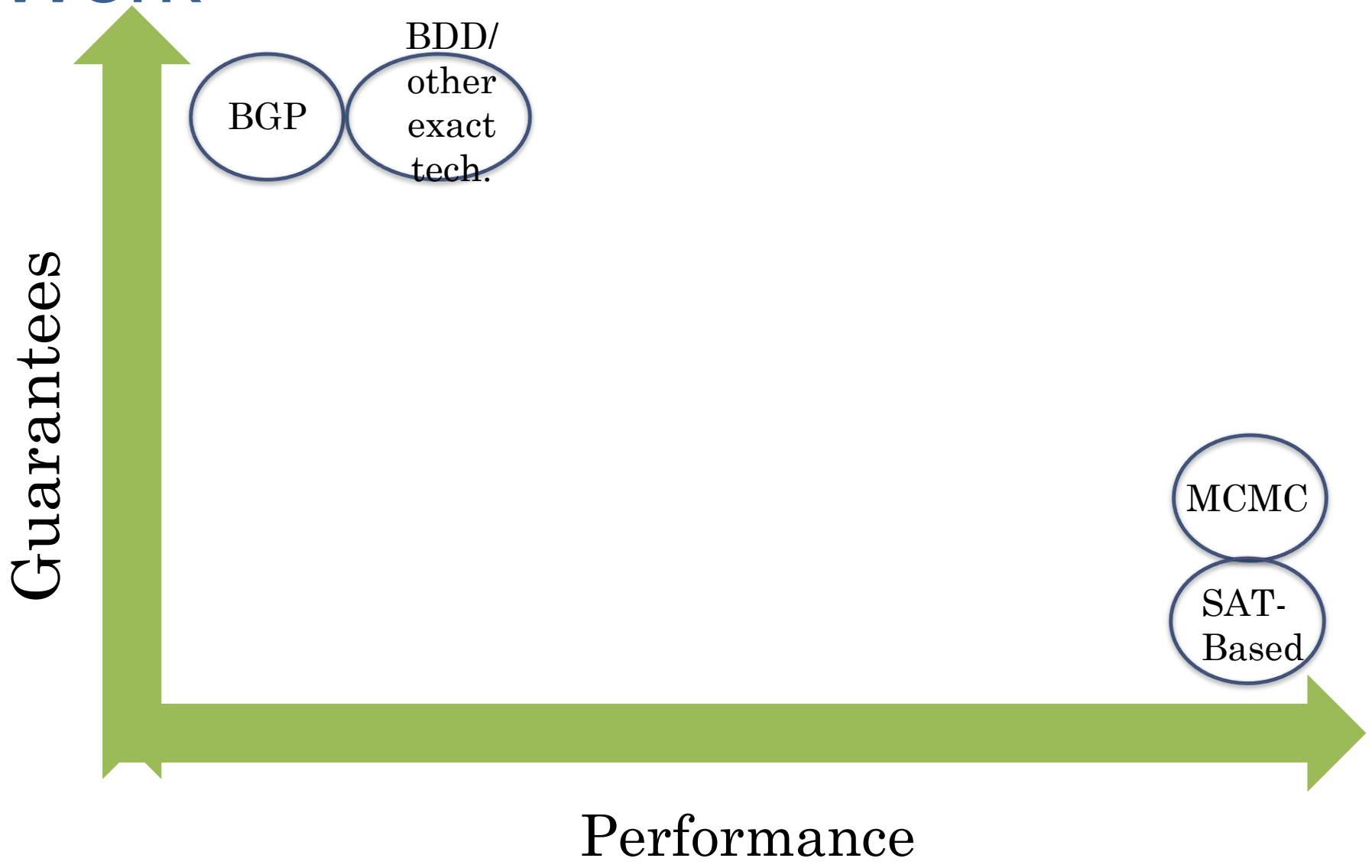
Approximate Integration and Sampling: Close Cousins

- Seminal paper by Jerrum, Valiant, Vazirani 1986



- Yet, no practical algorithms that scale to large problem instances were derived from this work
  - No scalable PAC counter or almost-uniform generator existed until a few years back
  - The inter-reductions are practically computation intensive
    - Think of O(n) calls to the counter when n = 100000

Prior Work


BGP
BDD/ other exact tech.
MCMC
SAT-Based
Guarantees
Performance

Techniques using XOR hash functions

- Bounding counters MBound, SampleCount [Gomes et al. 2006, Gomes et al 2007] used random XORs
  - Algorithms geared towards finding bounds without approximation guarantees
  - Power of 2-universal hashing not exploited
- In a series of papers [2013: ICML, UAI, NIPS; 2014: ICML; 2015: ICML, UAI; 2016: AAAI, ICML, AISTATS, …] Ermon et al used XOR hash functions for discrete counting/sampling
  - Random XORs, also XOR constraints with specific structures
  - 2-universality exploited to provide improved guarantees
  - Relaxed constraints (like short XORs) and their effects studied

An Interesting Combination: XOR + MAP Optimization

- WISH: Ermon et al 2013
- Given a weight function W: $\{0,1\}^n \rightarrow \Re^{\geq 0}$
  - Use random XORs to partition solutions into cells
  - After partitioning into 2, 4, 8, 16, … cells
    Use **Max Aposteriori Probability (MAP)** optimizer to find solution with max weight in a cell (say, a_2, a_4, a_8, a_{16}, …)
  - Estimated $W(R_F) = W(a_2)*1 + W(a_4)*2 + W(a_8)* 4 + \ldots$
- Constant factor approximation of $W(R_F)$ with high confidence
- MAP oracle needs repeated invokation $O(n.\log_2 n)$
  - MAP is NP-complete
  - Being optimization (not decision) problem), MAP is harder to solve in practice than SAT

XOR-based Counting Sampling

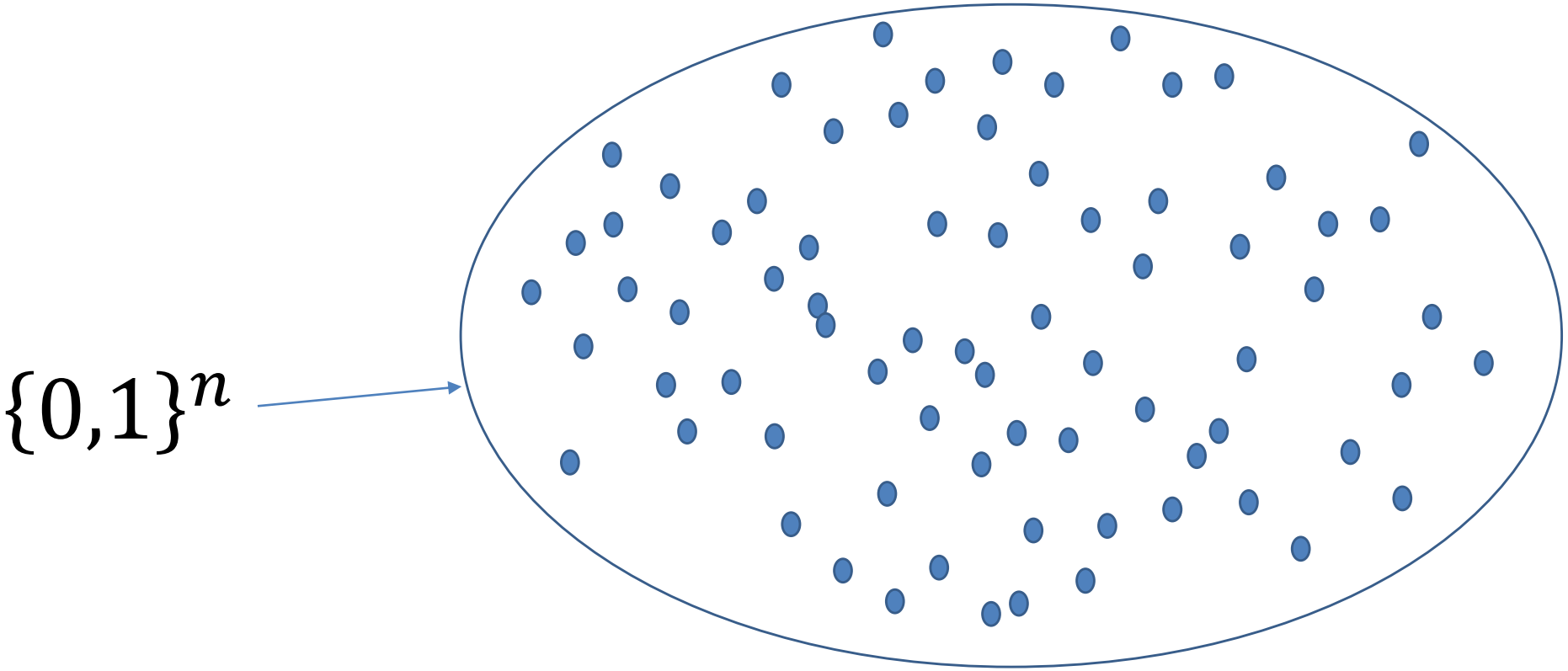
- Remainder of tutorial
  - Deeper dive into XOR hash-based counting and sampling
  - Discuss theoretical aspects and experimental observations
  - Leverage power of modern SAT solvers for CNF + XOR clauses (CryptoMiniSAT)
  - Based on work published in [2013: CP, CAV; 2014: DAC, AAAI; 2015: IJCAI, TACAS; 2016: AAAI, IJCAI, …]
  - Tutorial to focus mostly on unweighted case, to elucidate key ideas

Agenda (Part II)

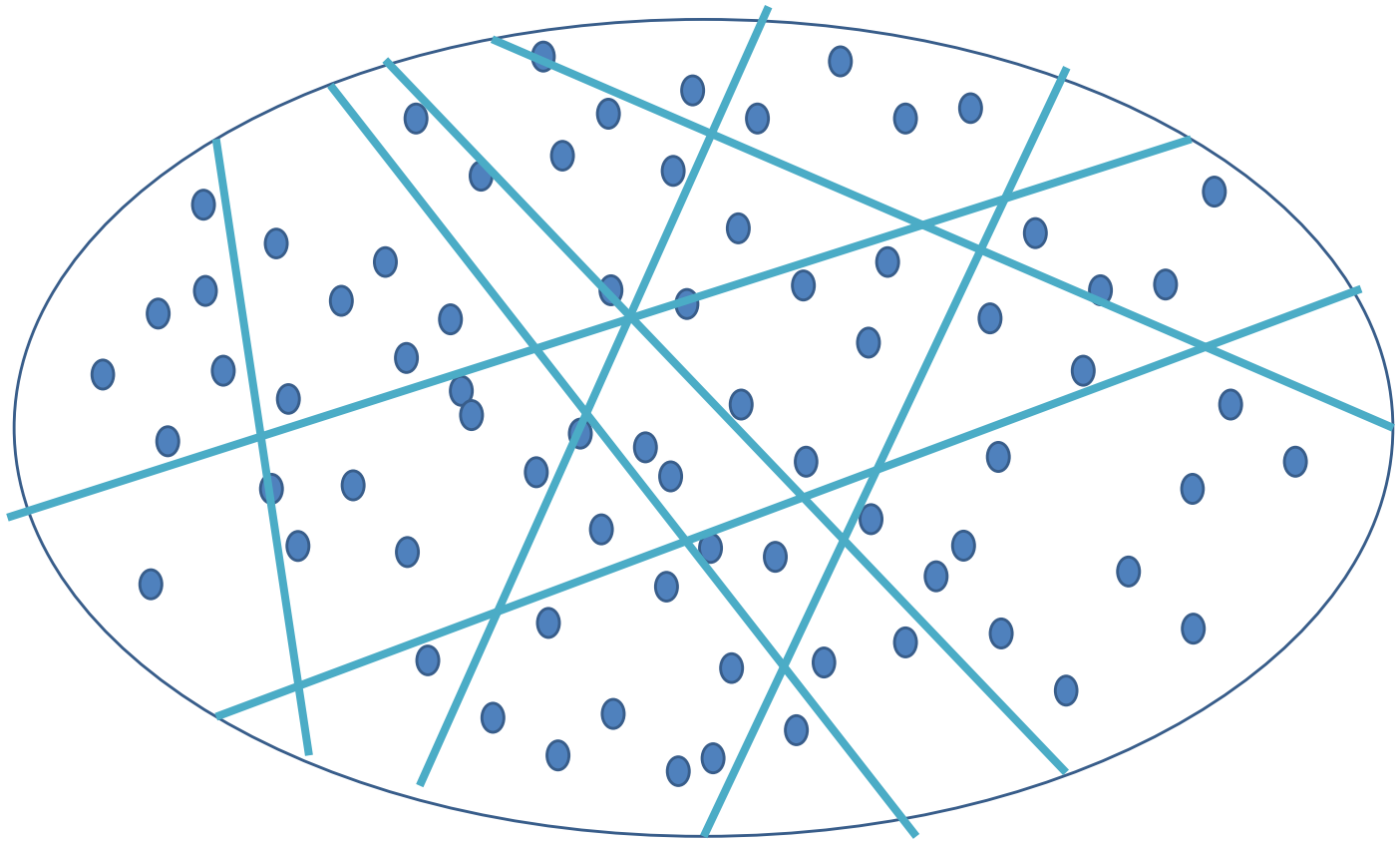
1. Hashing-based Approaches to Unweighted Model COunting
2. Hashing-based Approaches to Sampling
3. Design of Efficient Hash Functions
4. Summary

Counting Dots

- Solution to constraints

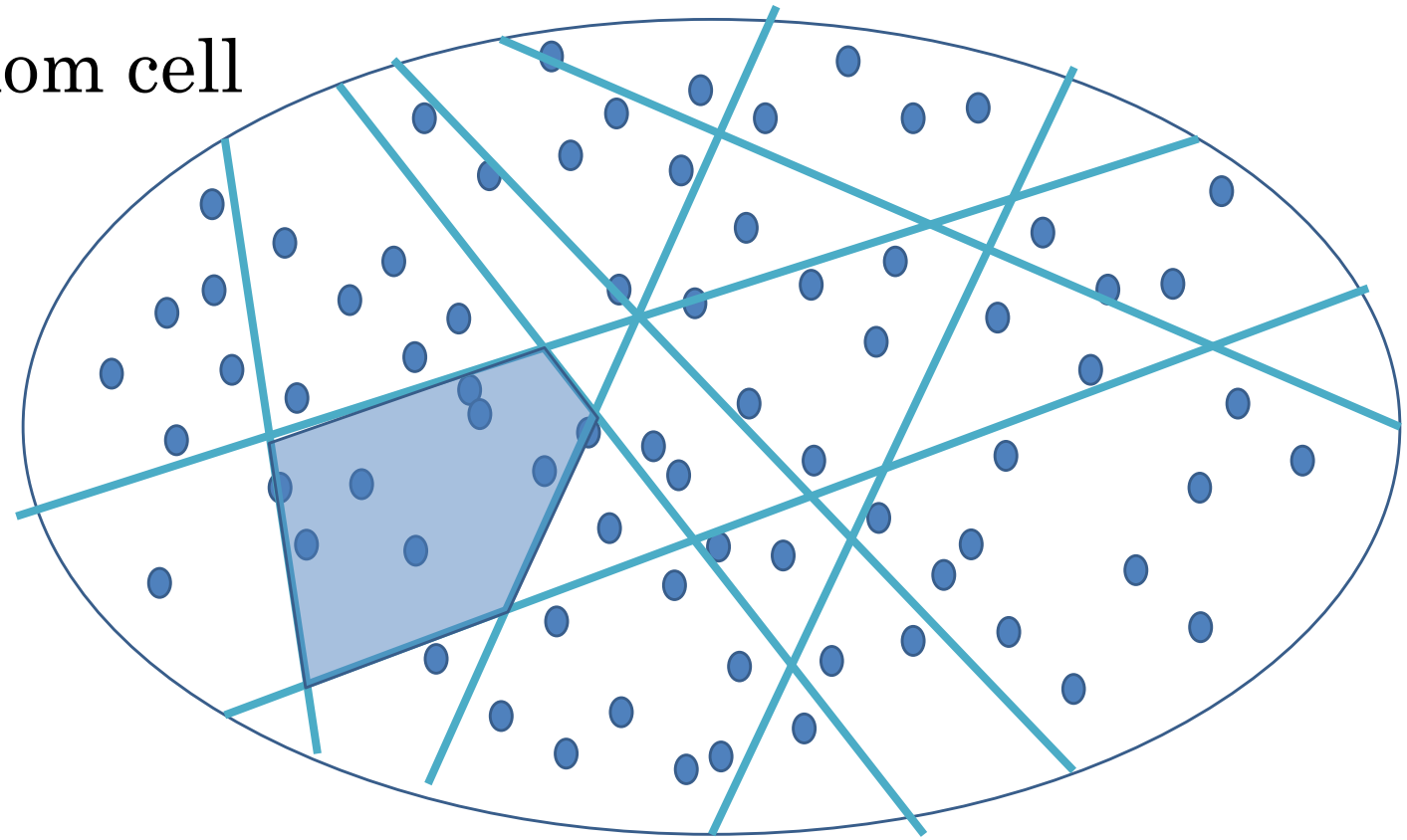


Partitioning into equal “small” cells



Partitioning into equal “small” cells

Pick a random cell



Estimate = # of solutions (dots) in cell * # of cells

How to Partition?

How to partition into roughly equal small cells of solutions without knowing the distribution of solutions?

2-Universal Hashing
[Carter-Wegman 1977]

Partitioning

1. How large is the “small” cell?

2. How do we compute solutions inside a cell?

3. How many cells?

Question 1: Size of cell

- Too large $\Rightarrow$ Hard to enumerate
- Too small $\Rightarrow$ Ratio of variance to mean is very high

$$pivot = 5\left(1 + \frac{1}{\varepsilon^2}\right);$$

Question 2: Solving a cell

- Variables: $X_1, X_2, X_3, \ldots, X_n$
- To construct h: $\{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- Pick every variable with prob. ½ , XOR them and add 1 with prob. ½
- E.g.: $X_1 \oplus X_3 \oplus X_6 \oplus \ldots \oplus X_{n-1}$
- $\alpha \in \{0,1\}^m \rightarrow$ Set every XOR equation to 0 or 1 randomly
- The cell: F $\wedge$ XOR (CNF+XOR)

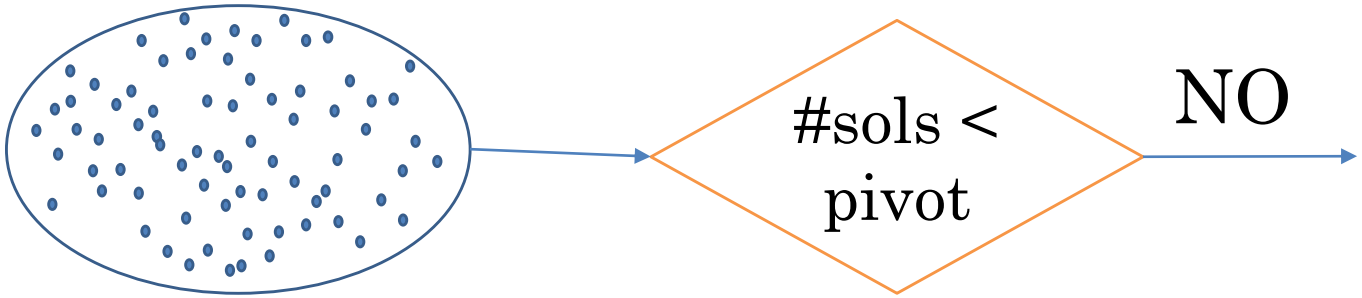
$$
\begin{aligned}
&X_1 \oplus X_3 \oplus X_6 \oplus \ldots X_{n-1} = 0 \\
&X_1 \oplus X_2 \oplus X_4 \oplus \ldots X_{n-1} = 1 \\
&X_1 \oplus X_3 \oplus X_5 \oplus \ldots X_{n-1} = 0 \\
&X_2 \oplus X_3 \oplus X_4 \oplus \ldots X_{n-1} = 0 \\
&\ldots\ldots \\
&X_1 \oplus X_2 \oplus X_3 \oplus \ldots X_{n-1} = 0
\end{aligned}
\quad \Big\} \; m \text{ XORs}
$$

Question 3: How many cells?

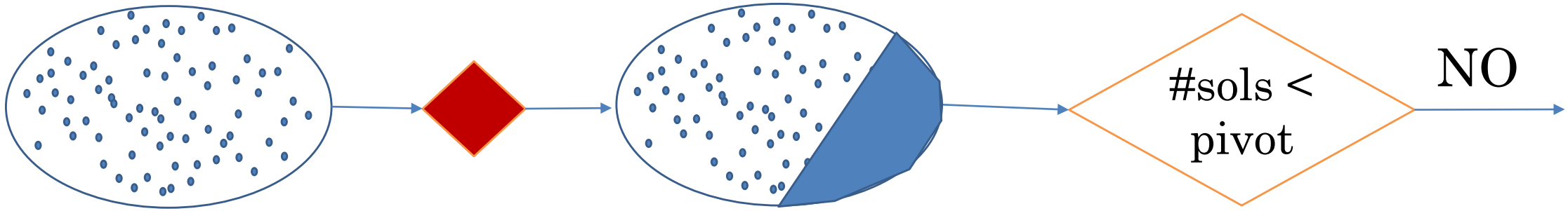
- We want to partition into 2^{m^*} cells such that $2^{m^*} = \frac{|R_F|}{pivot}$
  - Check for every m = 0,1….n if the number of solutions < pivot (function of ε)
  - Stop at the first m where number of solutions < pivot
  - Hash functions must be independent across different checks
- # of SAT calls is O(n)

(CP 2013)

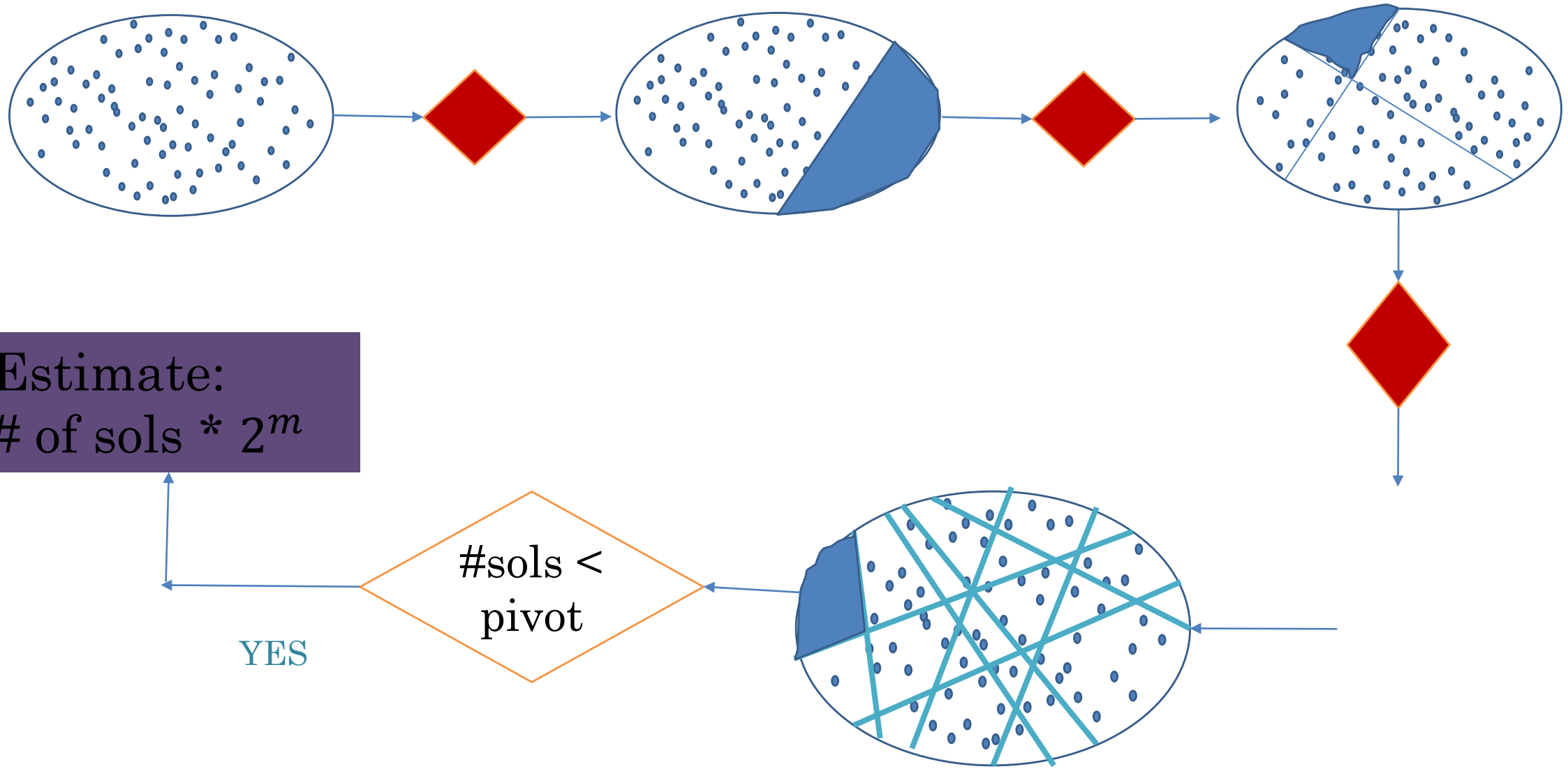
ApproxMC(F,ε, δ)



ApproxMC(F,ε, δ)



ApproxMC(F,ε, δ)



ApproxMC(F,ε, δ)

Key Lemmas

Let $m^* = \log \frac{|R_F|}{pivot}$ (i. e. , $2^{m^*} = \frac{|R_F|}{pivot}$)

Lemma 1: The algorithm terminates with $m \in [m^* - 1\ , m^*]$ with high probability

Lemma 2: The estimate from a randomly picked cell for $m \in [m^* - 1\ , m^*]$ is correct with high probability

ApproxMC(F,ε, δ)

Theorem 1:

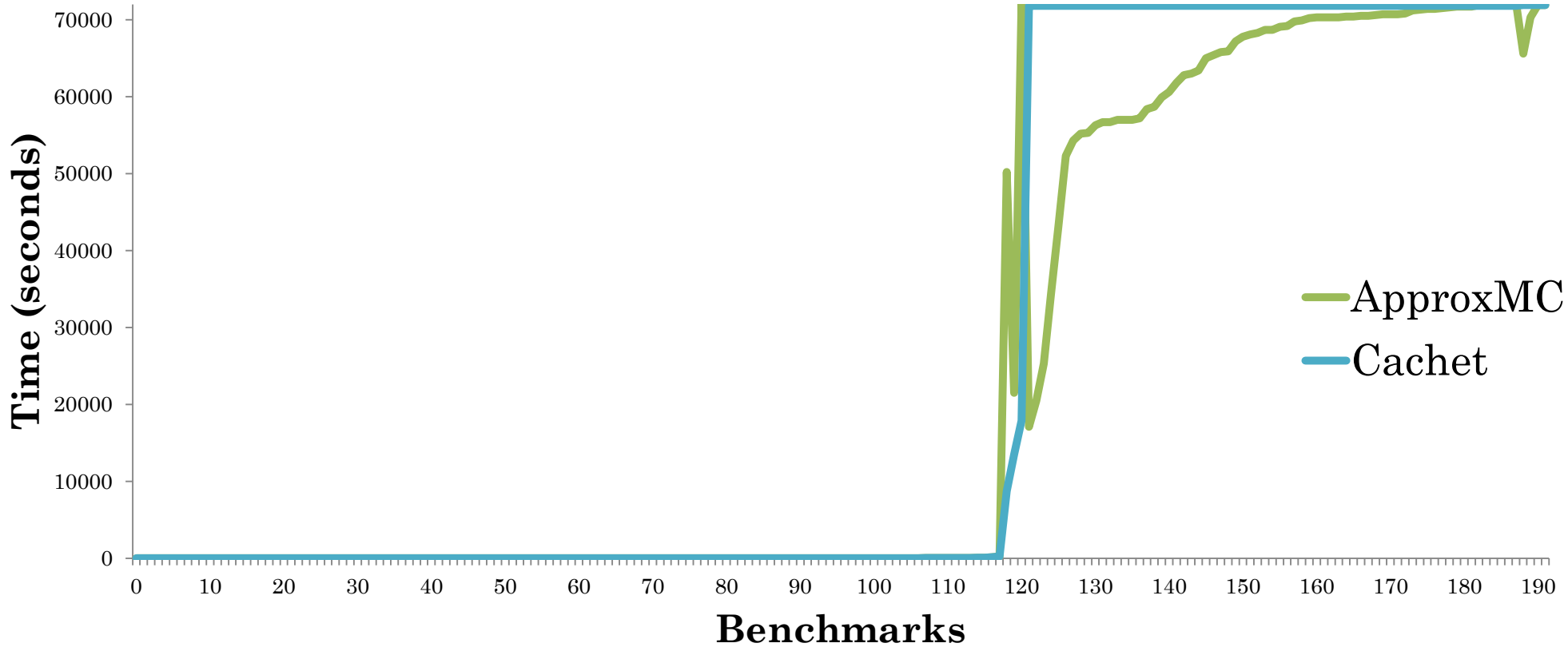
$$\Pr\left[\frac{|R_F|}{(1+\varepsilon)} \leq \text{ApproxMC(F},\varepsilon,\delta) \leq |R_F|(1+\varepsilon)\right] \geq 1-\delta$$

Theorem 2:

ApproxMC(F,ε, δ) makes $O\left(\frac{n \log\frac{1}{\delta}}{\varepsilon^2}\right)$ calls to NP oracle

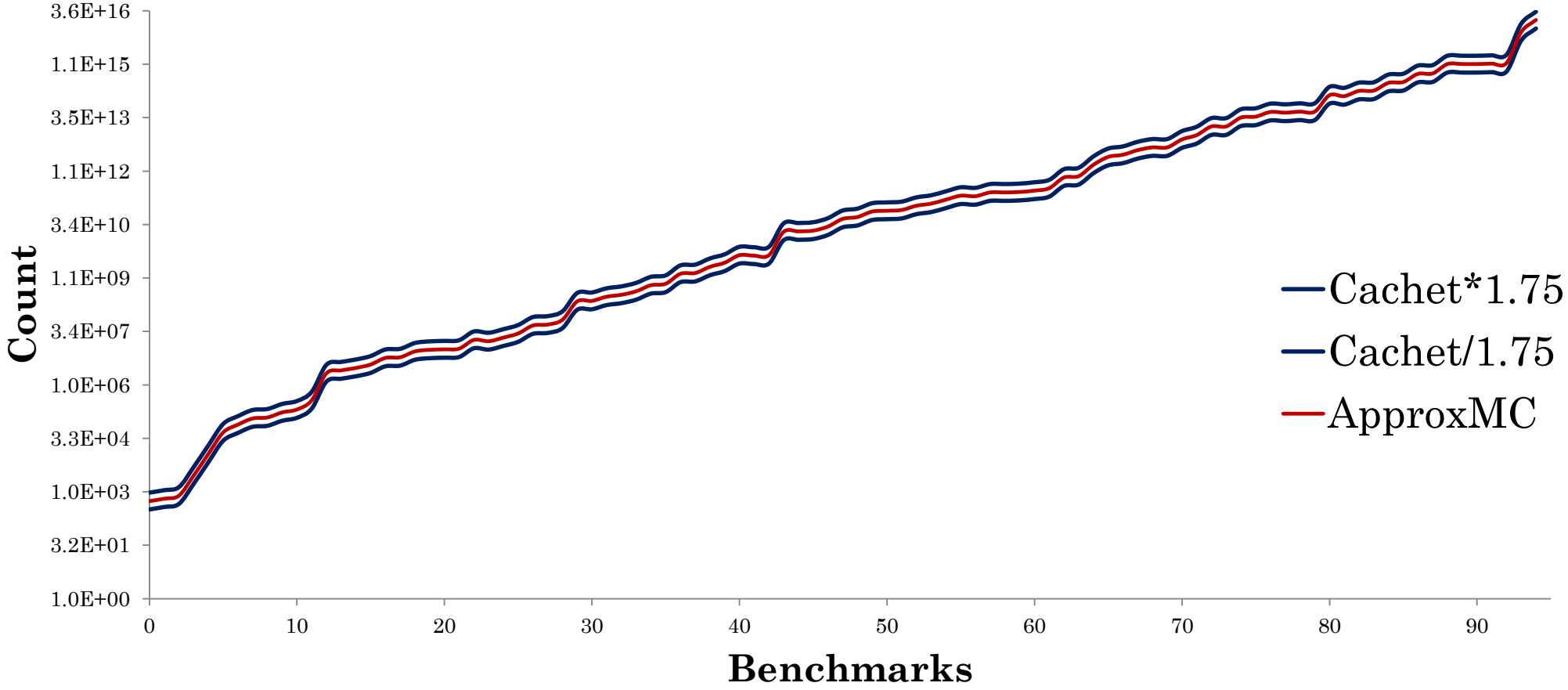
Runtime Performance of ApproxMC

Can Solve a Large Class of Problems



Large class of problems that lie beyond the exact algorithms but can be computed by ApproxMC

Mean Error: Only 4% (allowed: 75%)



Mean error: 4% – much smaller than the theoretical guarantee of 75%

Challenge

- Can we reduce the number of SAT calls from O(n)?

Experimental Observations

- ApproxMC "seems to work" even if we do not have independence across different hash functions
  - Can we really give up independence?

Beyond ApproxMC

- We want to partition into 2^m cells
  - Check for every m = 0,1….n if the number of solutions < pivot
  - Stop at the first m where number of solutions < pivot
  - Hash functions must be independent across different checks (Stockmeyer 1983, Jerrum, Valiant and Vazirani 1986…..)
- **Suppose:** Hash functions *can be dependent* across different checks
- # of solutions is monotonically non-increasing with m
  - Can find the right value of m by search in any order.
  - Binary search

ApproxMC2: From Linear to Logarithmic SAT calls

- The Proof: Hash functions *can be dependent* across different checks

- Key Idea: Probability of making a bad choice early on is very small.
  - Inversely (exponentially!) proportional to distance from m*)

(IJCAI 2016)

ApproxMC2(F,ε, δ)

Theorem 1:

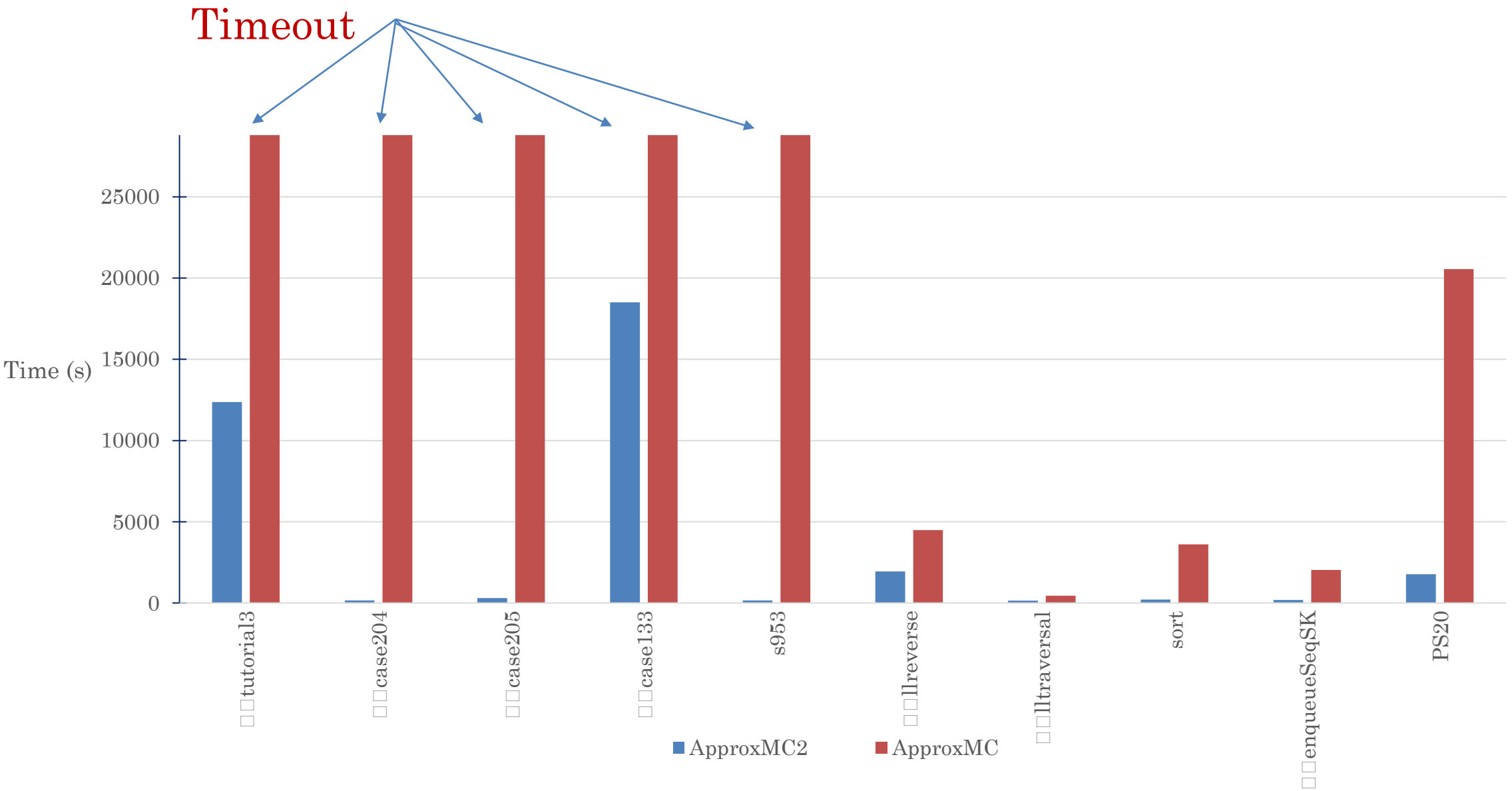
$$\Pr\left[\frac{|R_F|}{(1+\varepsilon)} \leq \text{ApproxMC2}(\text{F},\varepsilon,\delta) \leq |R_F|(1+\varepsilon)\right] \geq 1-\delta$$

Theorem 2:

ApproxMC2(F,ε, δ) makes $O\left(\frac{(\log n)\log\frac{1}{\delta}}{\varepsilon^2}\right)$ calls to NP oracle

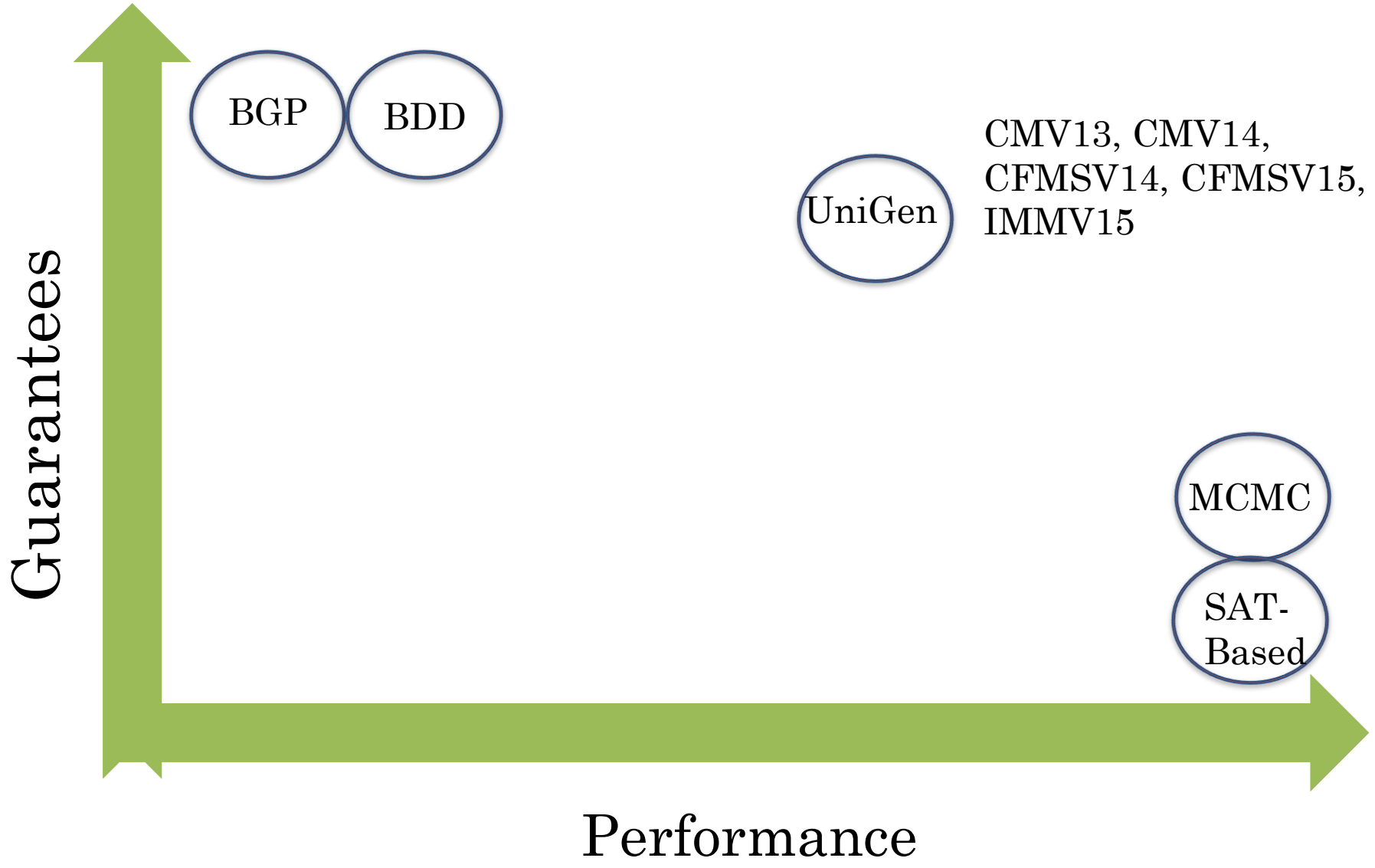
Theorem 1 requires a completely new proof.

Runtime Performance Comparison

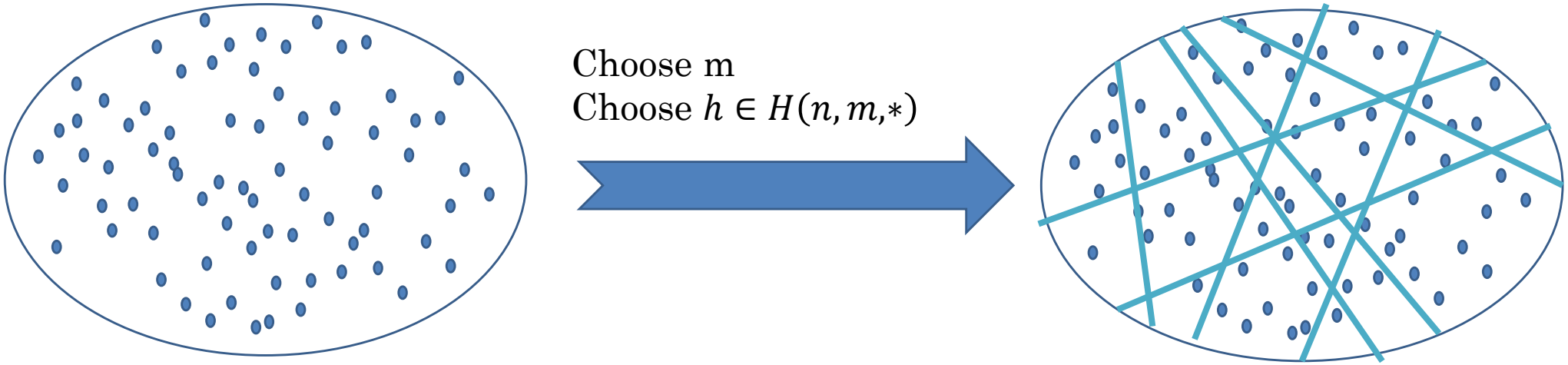


Discrete Uniform Sampling

Hashing-based Approaches



Key Ideas



- For right choice of m, large number of cells are “small”
  - “almost all” the cells are “roughly” equal
- Check if a randomly picked cell is “small”
- If yes, pick a solution randomly from randomly picked cell

Key Challenges

- F: Formula X: Set of variables R_F: Solution space
- $R_{F,h,\alpha}$: Set of solutions for $F \wedge (h(X) = \alpha)$ where
  - $h \in H(n, m, *)$; $\alpha \in \{0,1\}^m$

1. How large is “small” cell ?
2. How much universality do we need?
3. What is the value of m?

Size of cell

$$pivot = 5\left(1 + \frac{1}{\varepsilon^2}\right);$$

Independence

Theorem (CMV 14):

3-universal hashing is sufficient to provide almost uniformity.

(3-universality of XOR-based hash functions due to Gomes et al.)

How many cells?

- Our desire: $m = \log \frac{|R_F|}{pivot}$ (Number of cells: 2^m)
  - But determining $|R_F|$ is expensive (#P complete)

- How about approximation?
  - $ApproxMC(F, \varepsilon, \delta)$ returns C:

$$\Pr[\ \frac{|R_F|}{1+\varepsilon} \leq C \leq (1+\varepsilon)|R_F|] \geq 1 - \delta$$

  - $q = \log \frac{C}{pivot}$

  - Concentrate on m = q-1, q, q+1

UniGen(F,ε)

1. C = ApproxMC(F,ε)
2. Compute pivot
3. $q = \log|C| - \log pivot$
4. for i in {q-1, q, q+1}:
5. Choose h **randomly** from H(n,i,3)
6. Choose α randomly from $\{0,1\}^m$
7. If $(1 \leq |R_{F,h,\alpha}| \leq pivot)$:
8. Pick $y \in R_{F,h,\alpha}$ randomly

One time execution

Run for every sample required

Are we back to JVV (Jerrum, Valiant and Vazirani)?

NOT Really

- JVV makes linear (in n) calls to Approximate counter compared to just 1 in UniGen

- # of calls to ApproxMC is only 1 regardless of the number of samples required unlike JVV

Theoretical Guarantees

- Almost-Uniformity

  For every solution $y \in R_F$

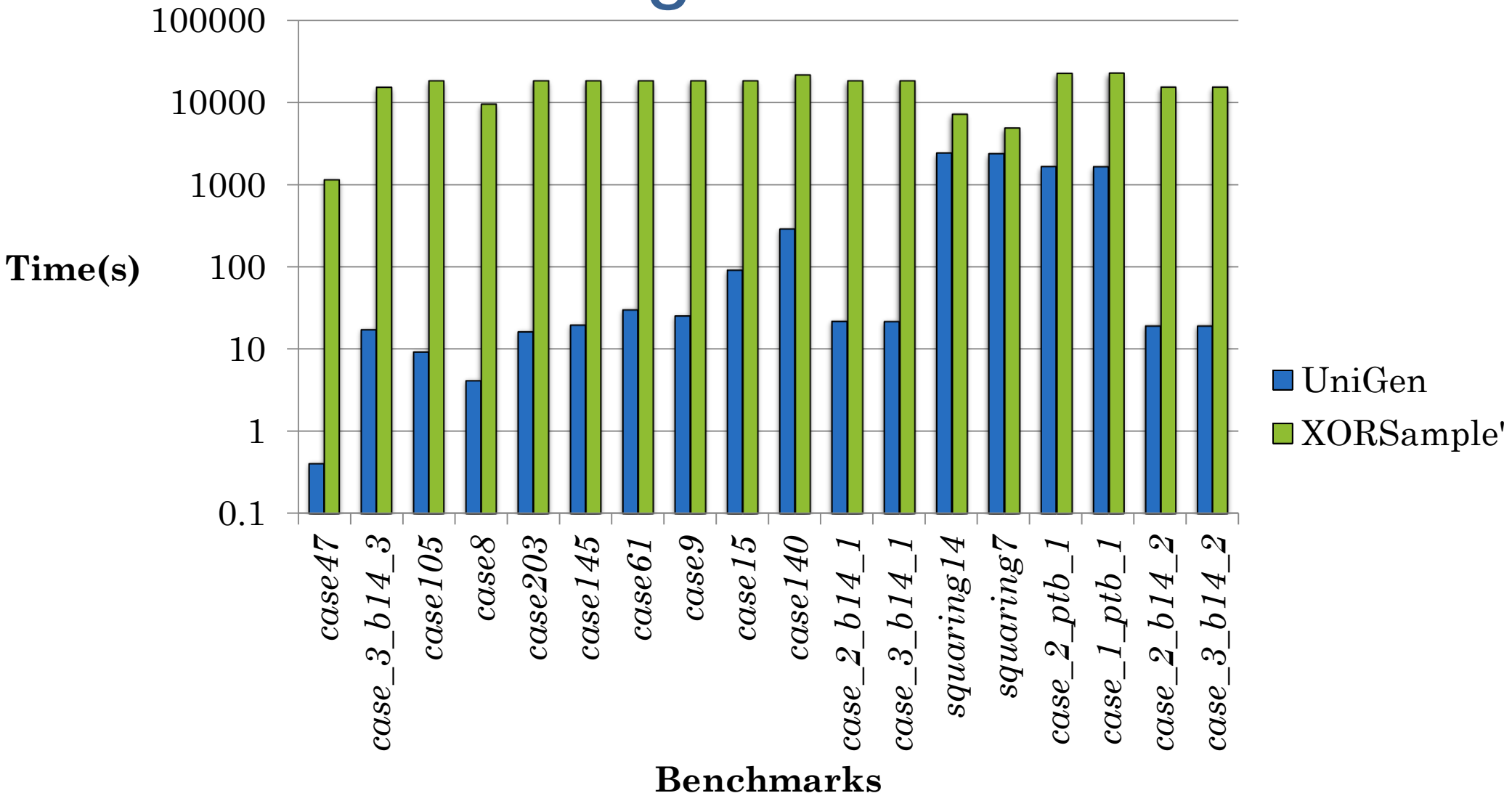
$$\forall y \in R_F, \quad \frac{1}{(1+\varepsilon)|R_F|} \leq \Pr[y \text{ is output}] \leq \frac{(1+\varepsilon)}{|R_F|}$$

- UniGen succeeds with probability ≥ 0.52
  - In practice, success probabiliy ≥ 0.99

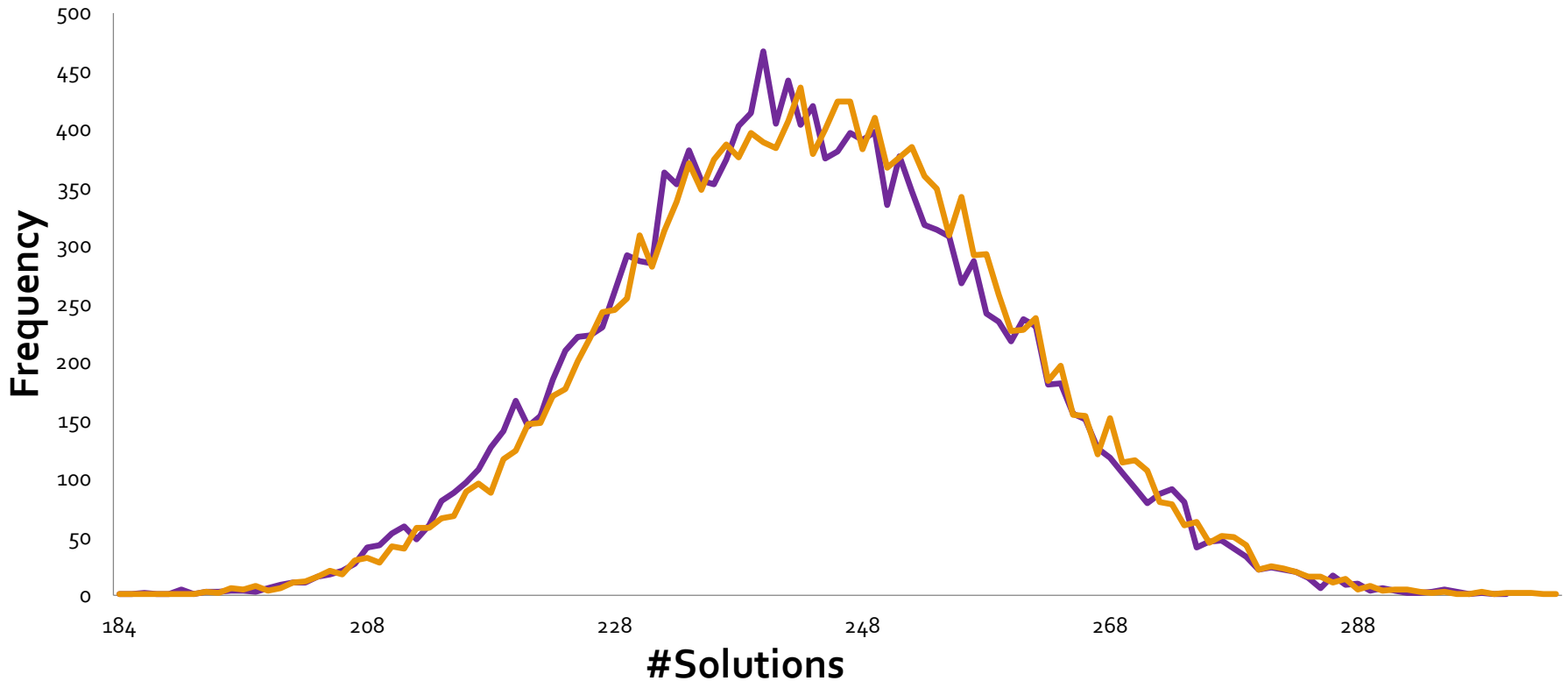
- UniGen makes $O(\frac{n}{\varepsilon^2})$ calls to NP oracle (SAT solver)

Runtime Performance of UniGen

1-2 Orders of Magnitude Faster

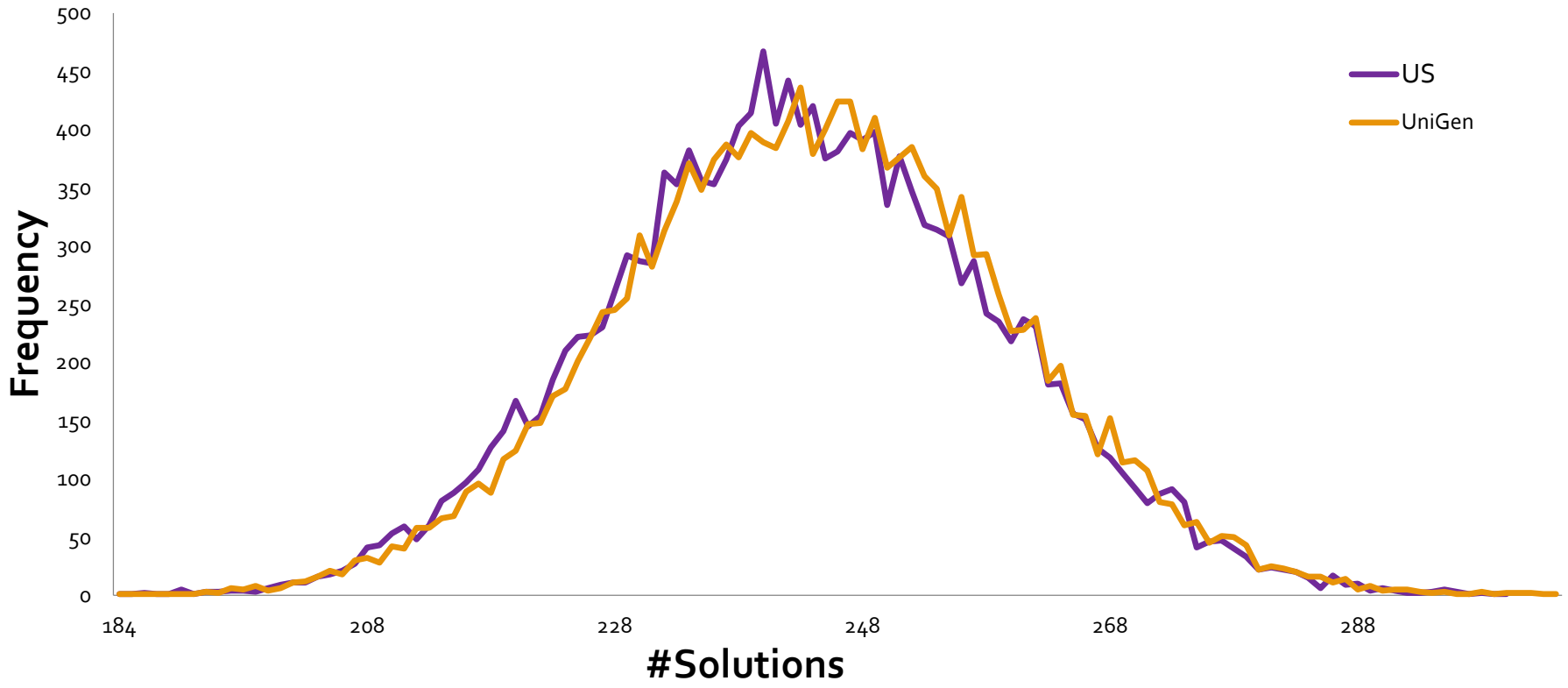


Results: Uniformity



- Benchmark: case110.cnf; #var: 287; #clauses: 1263
- Total Runs: $4x10^6$; Total Solutions : 16384

Results: Uniformity



- Benchmark: case110.cnf; #var: 287; #clauses: 1263
- Total Runs: $4x10^6$; Total Solutions : 16384

Contribution of Hashing-based Approaches

- ApproxMC: The first scalable approximate model counter
- UniGen: The first scalable uniform generator
- Outperforms state-of-the-art generators/counters

Towards Efficient Hash Functions

Parity-Based Hashing

- Variables: $X_1, X_2, X_3, \ldots\ldots, X_n$
- To construct h: $\{0,1\}^n \rightarrow \{0,1\}^m$, choose m random XORs
- Pick every variable with prob. ½ , XOR them and add 1 with prob. ½
- E.g.: $X_1 \oplus X_3 \oplus X_6 \oplus \ldots. \oplus X_{n-1}$
- $\alpha \in \{0,1\}^m \rightarrow$ Set every XOR equation to 0 or 1 randomly
- The cell: F $\wedge$ XOR (CNF+XOR)

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&X_2 \oplus X_3 \oplus X_4 \oplus \ldots. X_{n-1} = 0 \\
&\ldots\ldots \\
&X_1 \oplus X_2 \oplus X_3 \oplus \ldots. X_{n-1} = 0
\end{aligned}
\quad \left.\right\} \text{m XORs}
$$

Parity-Based Hashing

- Avg Length : n/2

- Smaller parity constraints➔ better performance

How to shorten XOR clauses?

Inspired from Error Correcting Codes

- X = # of solutions in a cell; $\mu_X = \frac{|R_F|}{2^m}$
- 2-universal hashing ensures $\sigma_X^2 \leq \mu_X$
- Key result: Using sparse constraints of size O(log n), we have:

$$\frac{\sigma_X^2}{\mu_X^2} \text{ is monotonically decreasing with X}$$

  - Challenge: Unable to guarantee $\sigma_X^2 \leq \mu_X$; therefore weaker concentration inequalities
- The resulting algorithms require $\theta(n \log n)$ NP calls in comparison to O(log n) calls based on 2-universal hashing algorithms

(Ermon et al 2014, 16; Achlioptas et al. 2015, Asteris et al 2016)

Independent Support

- Set I of variables such that assignments to these uniquely determine assignments to rest of variables (for satisfying assignments)
- If σ_1 and σ_2 agree on I then $\sigma_1 = \sigma_2$
- c $\leftrightarrow$ (a V b) ; Independent Support I: {a, b}
  - {a,c} is NOT an Independent Support
- ***Key Idea***: Hash only on the independent variables
  - Average size of XOR: $\frac{n}{2}$ to $\frac{|I|}{2}$

Formal Definition

Input Formula: F, Solution space: R_F

$\forall \sigma_1, \sigma_2 \in R_F$, If σ_1 and σ_2 agree on I, then $\sigma_1 = \sigma_2$

$$F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge \bigwedge_{i | x_i \in I} (x_i = y_i) \implies \bigwedge_j (x_j = y_j)$$

where $F(y_1, \ldots, y_n) = F(x_1 \to y_1, \ldots, x_n \to y_n)$

Key Idea

$$F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge \bigwedge_{i | x_i \in I} (x_i = y_i) \implies \bigwedge_j (x_j = y_j)$$

$$Q_{F,I} = F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge \bigwedge_{i | x_i \in I} (x_i = y_i) \wedge \neg \left(\bigwedge_j (x_j = y_j) \right).$$

Theorem: $Q_{F,I}$ is unsatisfiable if and only if I is independent support

Key Idea

$$H_1 = \{x_1 = y_1\},\ \ldots,\ H_n = \{x_n = y_n\}$$

$$\Omega = F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge (\neg \bigwedge_j (x_j = y_j))$$

$I = \{x_i\}$ is Independent Support iff $H^I \wedge \Omega$ is unsatisfiable where $H^I = \{H_i \,|x_i \in I\}$

Minimal Unsatisfiable Subset

- Given $\Psi = H_1 \wedge H_2 \cdots H_m \wedge \Omega$
  - Find subset $\{H_{i1}, H_{i2}, \cdots H_{ik}\}$ of $\{H_1, H_2, \cdots H_m\}$ such that $H_{i1} \wedge H_{i2} \cdots H_{ik} \wedge \Omega$ is UNSAT

    Unsatisfiable subset

  - Find **minimal** subset $\{H_{i1}, H_{i2}, \cdots H_{ik}\}$ of $\{H_1, H_2, \cdots H_m\}$ such that $H_{i1} \wedge H_{i2} \cdots H_{ik} \wedge \Omega$ is UNSAT

    Minimal Unsatisfiable subset

Minimal Independent Support

$$H_1 = \{x_1 = y_1\},\ \ldots,\ H_n = \{x_n = y_n\}$$

$$\Omega = F(x_1, \ldots, x_n) \wedge F(y_1, \ldots, y_n) \wedge (\neg \bigwedge_j (x_j = y_j))$$

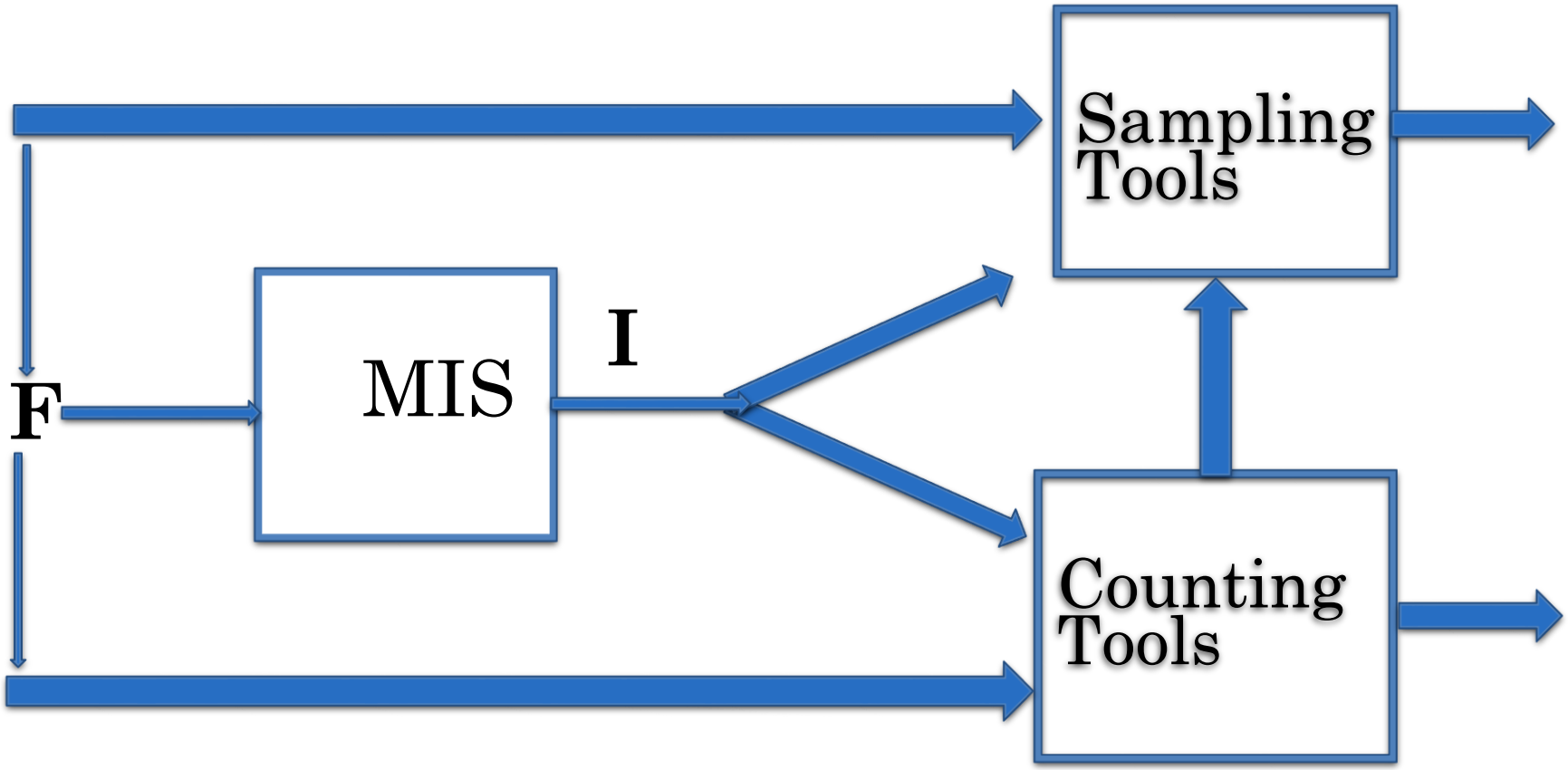
$I = \{x_i\}$ is minimal Independent Support iff H^I is minimal unsatisfiable subset where $H^I = \{H_i \,|x_i \in I\}$

Key Idea

Minimal Independent Support (MIS)

Minimal Unsatisfiable Subset (MUS)

Impact on Sampling and Counting Techniques



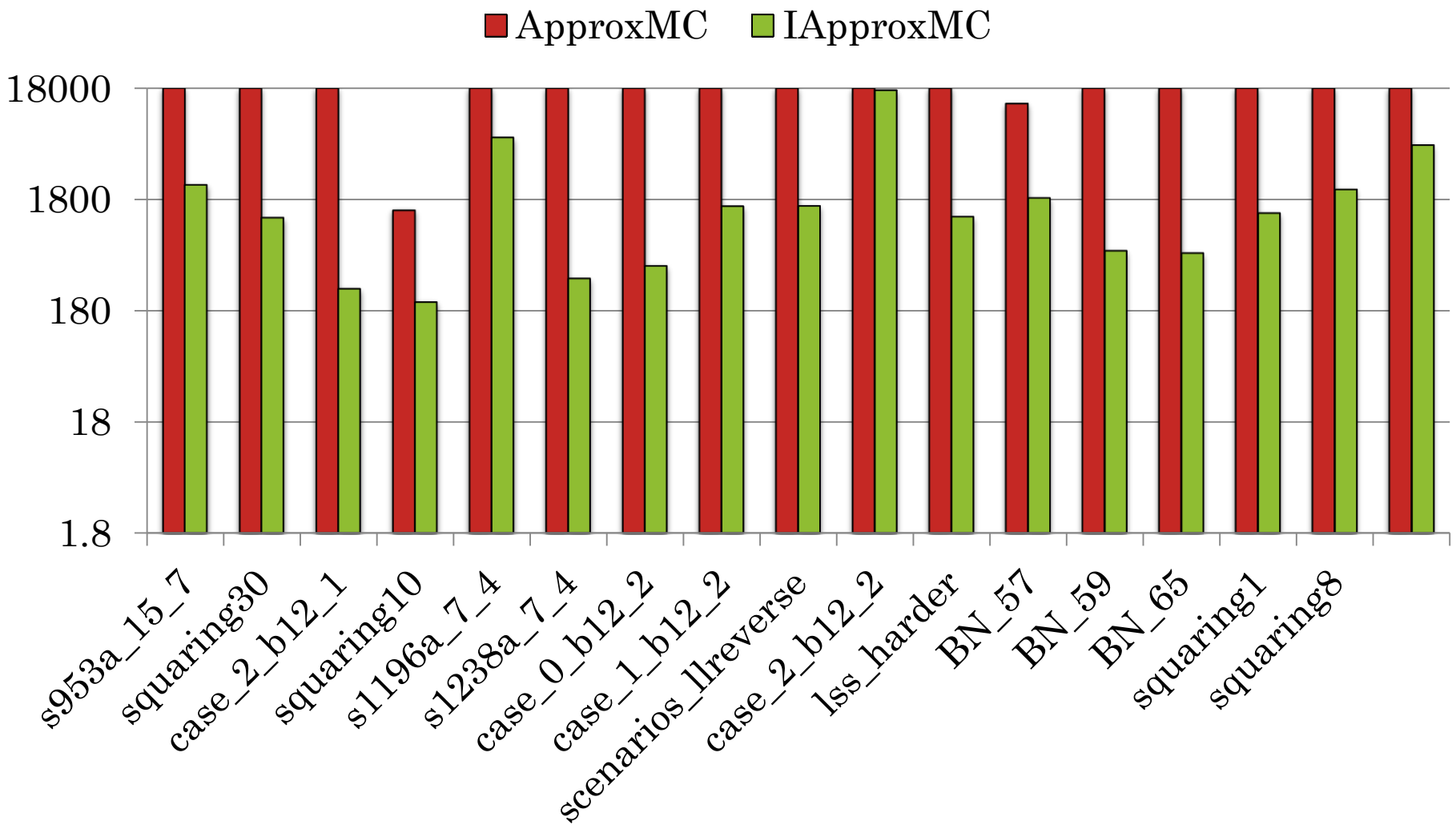
What about complexity

- Computation of MUS: FP^{NP}

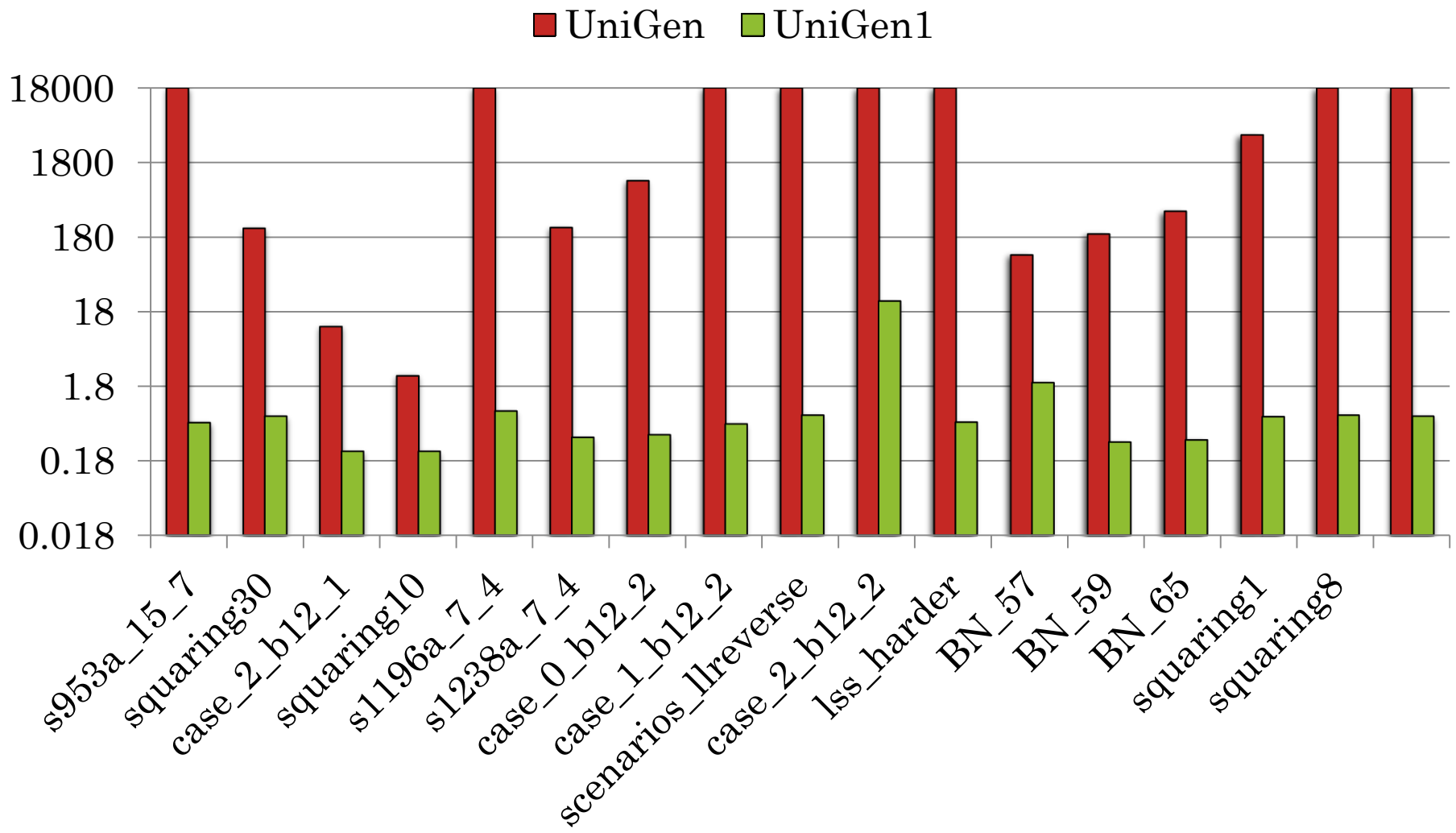
- Why solve a FP^{NP} for almost-uniform generation/approximate counter (PTIME PTM with NP Oracle)

Settling the debate through practice!

Performance Impact on Integration



Performance Impact on Uniform Sampling



Future Directions

Extension to More Expressive domains

- Efficient hashing schemes
  - Extending bit-wise XOR to richer constraint domains provides guarantees but fails to harness progress in solving engines for richer domains
- Solvers to handle F + Hash efficiently
  - CryptoMiniSAT has fueled progress for SAT domain
  - Similar solvers for other domains?
- Initial forays with bit-vector constraints and Boolector [AAAI 2016]
  - Uses new linear modular hash function that generalizes XOR-based hash functions
  - Significant speedups compared to bit-blasted versions

Summary

- Sampling and Integration are fundamental problems in Artificial Intelligence.
  - Applications from probabilistic inference, automatic problem generation to system verification.
- Drawback of related approaches: theoretical guarantees or scalability (Choose one)
- Hashing-based approaches promise theoretical guarantees and scalability

Take Away: Hashing-based Approaches

- Theoretical
  - Discrete Integration
    - Reduction of NP calls from O(n log n) to O(log n)
    - Efficient hash functions based on Independent support
  - Sampling
    - Reduction of Approximate Counting calls from O(n) to O(1)
    - Usage of 2-universal hash functions
- Practical
  - From problems with tens of variables (before 2013) to hundreds of thousands of variables

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(UCB)

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(CMS)

Thanks!

Questions?

Software and papers are available at **http://tinyurl.com/uai16tutorial**